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Discrete projective minimal surfaces

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ABSTRACT

We propose a natural discretisation scheme for classical projective minimal surfaces. We follow the classical geometric characterisation and classification of projective minimal surfaces and introduce at each step canonical discrete models of the associated geometric notions and objects. Thus, we introduce discrete analogues of classical Lie quadrics and their envelopes and classify discrete projective minimal surfaces according to the cardinality of the class of envelopes. This leads to discrete versions of Godeaux-Rozet, Demoulin and Tzitzéica surfaces. The latter class of surfaces requires the introduction of certain discrete line congruences which may also be employed in the classification of discrete projective minimal surfaces. The classification scheme is based on the notion of discrete surfaces which are in asymptotic correspondence. In this context, we set down a discrete analogue of a classical theorem which states that an envelope (of the Lie quadrics) of a surface is in asymptotic correspondence with the surface if and only if the surface is either projective minimal or a Q surface. Accordingly, we present a geometric definition of discrete Q surfaces and their relatives, namely discrete counterparts of classical semi-Q, complex, doubly Q and doubly complex surfaces.

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1. Introduction

Projective geometry is arguably "The Queen of All Geometries". In 1872, this was recognised by Felix Klein whose pioneering and universally accepted Erlangen program revolutionised the way (differential) geometry was approached and treated. Thus, Klein [13] proposed that geometries should be classified in terms of groups of transformations acting on a space (of homogeneous coordinates) with projective geometry being associated with the most encompassing group, namely the group of projective collineations. Apart from placing Euclidean and affine geometry in this context, this approach gives rise to projective models of a diversity of geometries such as Lie, Möbius, Laguerre and Plücker line geometries.

Projective differential geometry was initiated by Wilczynski [24–26] representing the "American School" and later formulated in an invariant manner by the "Italian School" whose founding members were Fubini and Ĉech. Cartan is but one of a long list of distinguished geometers who were involved in the development of this subject with monographs by Fubini & Ĉech [9,10], Lane [14], Finikov [8] and, most notably, Bol whose first two volumes [3,4] consist of more than 700 pages. Recently, there has been a resurgence of interest in projective differential geometry. Here, we mention the monograph *Projective Differential Geometry. Old and New* by Ovsienko and Tabachnikov [17] and the *Notes on Projective Differential Geometry* by Eastwood [5]. Interestingly, as pointed out in [18], projective differential geometry also finds application in General Relativity in connection with geodesic conservation laws.

Projective geometry has proven to play a central role in both discrete differential geometry and discrete integrable systems theory [2]. Thus, it turns out that classical incidence theorems of projective geometry such as Desargues', Möbius' and Pascal's theorems lie at the heart of the integrable structure prevalent in discrete differential geometry. Their algebraic incarnations provide the origin of the integrability of the associated discrete integrable systems such as the master Hirota (dKP), Miwa (dBKP) and dCKP equations (see [12,21] and references therein). A survey of this important subject may be found in the monograph [2]. Moreover, classical projective differential geometry has been shown to constitute a rich source of integrable geometries and associated nonlinear differential equations [6,7].

Here, in view of the development of a canonical discrete analogue of projective differential geometry, we are concerned with the important class of projective minimal surfaces [4]. These have a great variety of geometric and algebraic properties and are therefore custom made for the identification of a general discretisaton procedure which preserves essential geometric and algebraic features. In this connection, it should be pointed out that projective minimal surfaces have been shown to be integrable (see [7,19] and references therein) in the sense that the underlying projective "Gauss–Mainardi–Codazzi equations" are amenable to the powerful techniques of integrable systems theory (soliton theory) [19]. It turns out that the discretisation scheme proposed here preserves integrability and even though this important aspect will be discussed in a separate publication