Advances in Mathematics 336 (2018) 242–298

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Thick subcategories of discrete derived categories

Nathan T. Broomhead

School of Electronics, Computing and Mathematics, Plymouth University, Plymouth, PL4 8AA, UK

A R T I C L E I N F O

Article history:
Received 13 September 2017
Received in revised form 12 June 2018
Accepted 24 July 2018
Available online xxxx
Communicated by Henning Krause

Keywords:
Discrete derived category
Thick subcategory
Geometric model
Exceptional collection
ES-collection
Mutation

A B S T R A C T

We classify the thick subcategories of discrete derived categories. To do this we introduce certain generating sets called ES-collections which correspond to configurations of non-crossing arcs on a geometric model. We show that every thick subcategory is generated by an ES-collection and we describe a version of mutation which acts transitively on the set of ES-collections generating a given thick subcategory.

Contents

0. Introduction . . . 243
1. Background on discrete derived categories and calculation of cones . . . 246
2. Generators for thick subcategories . . . 252
3. A geometric model . . . 254
4. ES-collections . . . 269
5. Morphisms in ES-collections and factoring arcs . . . 273
6. Reduced collections . . . 275
7. Mutation of ES-collections . . . 279
8. The lattice of thick subcategories . . . 284

E-mail address: nathan.broomhead@plymouth.ac.uk.

https://doi.org/10.1016/j.aim.2018.07.029

Appendix A. The Grothendieck group 295
References 297

0. Introduction

In this article, we study thick subcategories of discrete derived categories $\mathsf{D}^b(\mathsf{mod}(\Lambda))$ (introduced by Vossieck in [27]), in the case where Λ has finite global dimension and is not of derived-finite representation type. By a *thick* subcategory, we mean a triangulated subcategory which is closed under taking direct summands. The set of all thick subcategories of any essentially small triangulated category forms a lattice with respect to the partial order given by inclusion, and this is an interesting invariant of the category.

The study of thick subcategories has a long history and descriptions of the lattice exist in the literature in various contexts. For example, Devinatz, Hopkins and Smith treated certain stable homotopy categories [11,17], Hopkins [16] and Neeman [23] considered the category of perfect complexes over a commutative noetherian ring, and Thomason [26] generalised this to perfect complexes over quasi-compact quasi-separated schemes. More recently there has been further interesting work by many authors; see for example [3,4,24]. However, all of these results depend in some way on having a tensor structure, and without such a structure the set of examples is more limited.

If A is a finite dimensional algebra over a field (or more generally for certain connected hereditary Artin algebras), then there is a classification of the thick subcategories of $\mathsf{D}^b(\mathsf{mod}(A))$ which are generated by exceptional collections. This was due to Ingalls and Thomas [20] for the path algebra of Dynkin or extended Dynkin type, and generalised by Igusa, Schiffler, Krause, Hubery, Antieau and Stevenson [19,22,18,1]. In the case when the algebra A is of Dynkin type, all thick subcategories are generated by exceptional collections, and so this forms a complete classification.

By considering the thick subcategories generated by exceptional collections and those generated by regular objects, there is also a classification of thick subcategories of derived categories of tame hereditary algebras (see work of Brüning, Dichev and Köhler [9,12, 21]). However, the two types of thick subcategories are not treated in a uniform way, and so this classification doesn't yield the lattice structure. Finally, Takahashi gave a classification of the thick subcategories of the singularity categories of local hypersurfaces, which didn't use a tensor structure [25].

In this paper we add to this rather short list, a three parameter family of examples for which the classification is complete, and the lattice structure is understood. Discrete derived categories form a class of triangulated categories which are complicated enough to exhibit many interesting properties but at the same time are tractable enough that calculations can be done explicitly. Their structure has been studied by Bobiński, Geiß, and Skowroński [5] and more recently by the author with Pauksztello and Ploog [8,7], and this detailed understanding underpins the proofs of the classification in this paper. I believe however, that many of the techniques will in fact generalise to bounded de-