Advances in Mathematics 336 (2018) 316–334

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

New Pólya–Szegö-type inequalities and an alternative approach to comparison results for PDE's

F. Brock [a], F. Chiacchio [c], A. Ferone [b], A. Mercaldo [c,*]

[a] ISAM, South China University of Technology, Linhong Bldg. Wushan Campus 381 Wushan Road, Tianhe District, 510641, Guangzhou, PR China
[b] Dipartimento di Matematica, Università degli Studi della Campania Luigi Vanvitelli, Viale Lincoln 5, 81100 Caserta, Italy
[c] Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli Federico II, Complesso Monte S. Angelo, via Cintia, 80126 Napoli, Italy

A R T I C L E   I N F O

Article history:
Received 6 April 2017
Received in revised form 2 February 2018
Accepted 15 July 2018
Available online xxxx
Communicated by Gang Tian

MSC:
26D15
35J15
35J25

Keywords:
Pólya–Szegö principle
Steiner symmetrization
Elliptic equations
Comparison results

A B S T R A C T

We prove some Pólya–Szegö type inequalities which involve couples of functions and their rearrangements. Our inequalities reduce to the classical Pólya–Szegö principle when the two functions coincide. As an application, we give a different proof of a comparison result for solutions to Dirichlet boundary value problems for Laplacian equations proved in [1].

* Corresponding author.
E-mail addresses: fbrock@scut.edu.cn (F. Brock), francesco.chiacchio@unina.it (F. Chiacchio), adele.ferone@unicampania.it (A. Ferone), mercaldo@unina.it (A. Mercaldo).

https://doi.org/10.1016/j.aim.2018.07.026
0001-8708/

1. Introduction

The celebrated Pólya–Szegö Principle asserts that Dirichlet type integrals do not increase under Schwarz symmetrization. In its simplest form it states that, if u is a compactly supported function which belongs to $W^{1,2}(\mathbb{R}^N)$ then also its *spherically symmetric rearrangement* $u^\star$ is in $W^{1,2}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |\nabla u(z)|^2 dz \geq \int_{\mathbb{R}^N} |\nabla u^\star(z)|^2 dz\,. \tag{1.1}$$

The interest in the Pólya–Szegö principle is due to its multitude of applications in analysis and physics. For instance, it is the main tool in proving isoperimetric inequalities for capacities and of Faber–Krahn type, as well as apriori estimates for solutions to boundary value problems for PDEs (see e.g. [6], [27], [28], [29], [33] and references therein). The topic has attracted the attention of many authors, and it has been developed in various directions since the middle of last century. For instance, more general functionals of the gradient under different types of symmetrizations have been investigated (see, for example, [4], [9], [11], [13], [20], [22], [23], [32] and references therein). More recently, the equality case and the stability in these inequalities have been studied (see, for example, [12], [26]).

In this paper we prove a Pólya–Szegö type inequality which – unlike the classical case (1.1) – involves *two* functions u, w and their rearrangements. Our inequality reduces to (1.1) when $u = w$. We focus on the Steiner symmetrization, and we will analyze the differences which appear when Steiner symmetrization is replaced by Schwarz symmetrization.

The proofs of Pólya–Szegö type inequalities are typically based on the isoperimetric inequality in Euclidean space, while our approach relies on two further well-known tools from the theory of rearrangements: the *Hardy–Littlewood inequality* and the *Riesz inequality*. The Hardy–Littlewood inequality states that

$$\int_{\mathbb{R}^N} u(z)w(z)dz \leq \int_{\mathbb{R}^N} u^\#(z)w^\#(z)dz\,, \tag{1.2}$$

for any couple of measurable nonnegative functions. Throughout the present paper, a superscript hashtag # denotes Steiner rearrangement of both sets and functions – see Section 2 for relevant definitions.

The main result of the paper is

Theorem 1.1. *Assume that u and w are Lipschitz-continuous nonnegative functions with compact support defined in $\mathbb{R}^N$ and*

$$\int_{\mathbb{R}^N} u(z)w(z)dz = \int_{\mathbb{R}^N} u^\#(z)w^\#(z)dz\,. \tag{1.3}$$