

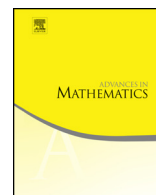


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## New Pólya–Szegő-type inequalities and an alternative approach to comparison results for PDE's

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### ABSTRACT

We prove some Pólya–Szegő type inequalities which involve couples of functions and their rearrangements. Our inequalities reduce to the classical Pólya–Szegő principle when the two functions coincide. As an application, we give a different proof of a comparison result for solutions to Dirichlet boundary value problems for Laplacian equations proved in [1].

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### 1. Introduction

The celebrated Pólya–Szegő Principle asserts that Dirichlet type integrals do not increase under Schwarz symmetrization. In its simplest form it states that, if  $u$  is a compactly supported function which belongs to  $W^{1,2}(\mathbb{R}^N)$  then also its *spherically symmetric rearrangement*  $u^*$  is in  $W^{1,2}(\mathbb{R}^N)$  and

$$\int_{\mathbb{R}^N} |\nabla u(z)|^2 dz \geq \int_{\mathbb{R}^N} |\nabla u^*(z)|^2 dz. \tag{1.1}$$

The interest in the Pólya–Szegő principle is due to its multitude of applications in analysis and physics. For instance, it is the main tool in proving isoperimetric inequalities for capacities and of Faber–Krahn type, as well as apriori estimates for solutions to boundary value problems for PDEs (see e.g. [6], [27], [28], [29], [33] and references therein). The topic has attracted the attention of many authors, and it has been developed in various directions since the middle of last century. For instance, more general functionals of the gradient under different types of symmetrizations have been investigated (see, for example, [4], [9], [11], [13], [20], [22], [23], [32] and references therein). More recently, the equality case and the stability in these inequalities have been studied (see, for example, [12], [26]).

In this paper we prove a Pólya–Szegő type inequality which – unlike the classical case (1.1) – involves *two* functions  $u, w$  and their rearrangements. Our inequality reduces to (1.1) when  $u = w$ . We focus on the Steiner symmetrization, and we will analyze the differences which appear when Steiner symmetrization is replaced by Schwarz symmetrization.

The proofs of Pólya–Szegő type inequalities are typically based on the isoperimetric inequality in Euclidean space, while our approach relies on two further well-known tools from the theory of rearrangements: the *Hardy–Littlewood inequality* and the *Riesz inequality*. The Hardy–Littlewood inequality states that

$$\int_{\mathbb{R}^N} u(z)w(z) dz \leq \int_{\mathbb{R}^N} u^\#(z)w^\#(z) dz, \tag{1.2}$$

for any couple of measurable nonnegative functions. Throughout the present paper, a superscript hashtag # denotes Steiner rearrangement of both sets and functions – see Section 2 for relevant definitions.

The main result of the paper is

**Theorem 1.1.** *Assume that  $u$  and  $w$  are Lipschitz-continuous nonnegative functions with compact support defined in  $\mathbb{R}^N$  and*

$$\int_{\mathbb{R}^N} u(z)w(z) dz = \int_{\mathbb{R}^N} u^\#(z)w^\#(z) dz. \tag{1.3}$$

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