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Petersson norms of not necessarily cuspidal Jacobi modular forms and applications

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ABSTRACT

We extend the usual notion of Petersson inner product on the space of cuspidal Jacobi forms to include non-cuspidal forms as well. This is done by examining carefully the relation between certain "growth-killing" invariant differential operators on $\mathbf{H}_2$ and those on $\mathbf{H}_1 \times \mathbf{C}$ (here $\mathbf{H}_n$ denotes the Siegel upper half space of degree n). As applications, we can understand better the growth of Petersson norms of Fourier Jacobi coefficients of Klingen Eisenstein series, which in turn has applications to finer issues about representation numbers of quadratic forms; and as a by-product we also show that *any* Siegel modular form of degree 2 is determined by its 'fundamental' Fourier coefficients.

1. Introduction

Non-cuspidal elliptic modular forms decompose into an Eisenstein series and a cusp form. This gives at the same time a decomposition of its Fourier coefficients into a

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dominant term (coming from the Eisenstein series) and an error term (coming from the cuspidal part). For Siegel modular forms of higher degrees the main obstacle to clean asymptotic properties of the Fourier coefficients are the so-called Klingen–Eisenstein series attached to cusp forms of lower degree. Their Fourier coefficients grow in a similar way as the Fourier coefficients of Siegel Eisenstein series (which are natural candidates for the dominant term), as long as we consider them indexed by matrices in certain subsets of half-integral symmetric and positive definite matrices. We refer the reader to [6,14] for a variety of results in this direction.

A first attempt to a better understanding of this phenomenon was made in [4] for the special case of degree 2. The key observation was to use the Fourier–Jacobi coefficients ϕ_m of the Klingen Eisenstein series, and to consider its decomposition into a Jacobi–Eisenstein part and a cuspidal part:

$$\phi_m = \mathscr{E}_{k,m} + \phi_m^o.$$

Vaguely speaking, the cuspidal part ϕ_m° behaves (almost) like the Fourier–Jacobi coefficient of a Siegel cusp form, whereas the $\mathscr{E}_{k,m}$'s are responsible for the dominating part of the Fourier coefficients of the Klingen Eisenstein series. These properties were shown in [4] for special types of exponent matrices T, e.g., those T for which $-\det(2T)$ is a fundamental discriminant. The basic tool in [4] was to identify the Fourier coefficients of the $\mathscr{E}_{k,m}$ with subseries of the infinite series giving the Fourier coefficients of the Klingen–Eisenstein series. This interplay was recently worked out in complete generality in the Ph.D. thesis of T. Paul [21,22].

One of the main purposes of the present paper is to get a better understanding of the growth properties of those cuspidal parts ϕ_m^o, in particular their Petersson norms; from this one then gets growth properties also for the Fourier coefficients. This is obtained in the form of an asymptotic formula, in Theorem 6.8. To do this we take a detour (mainly because the Dirichlet series $\sum_{m\geq 1}\langle \phi_m^\circ, \phi_m^\circ\rangle m^{-s}$ does not seem to have good analytic properties, see Remark 6.6), which is at the same time the second main topic of our paper. Namely using certain "growth killing" invariant differential operators, we define an extension $\{\,,\}$ of the classical Petersson product $\langle\,,\rangle$ to the full space of Jacobi forms. The idea is that we are able to estimate $\{\phi_m, \phi_m\}$ first and compute $\{\mathscr{E}_{k,m}, \mathscr{E}_{k,m}\}$ explicitly; thereby obtaining a bound for $\{\phi_m^\circ, \phi_m^\circ\}$. Note that this is proportional to the square of the standard Petersson norm for ϕ_m°, because ϕ_m° is cuspidal.

Before discussing the content of the paper in more detail, let us indicate a few applications of our construction and ideas involved in this paper. ***Throughout this paper, we assume that*** k ***is even.*** One of the reasons is that there is no reasonable way of introducing Eisenstein series for odd weights for the Siegel modular group (cf. [10, p. 63]).

Applications: Apart from the intrinsic interest in the extended Petersson norm, we can give the following applications.