Advances in Mathematics 336 (2018) 377–408

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Nilpotency in instanton homology, and the framed instanton homology of a surface times a circle

William Chen [a], Christopher Scaduto [b,*]

[a] *Ben-Gurion University, Mathematics Department, Beer-Sheva, Israel*
[b] *Simons Center for Geometry and Physics, Stony Brook, NY, United States of America*

ARTICLE INFO

Article history:
Received 31 December 2016
Received in revised form 15 May 2018
Accepted 20 July 2018
Available online xxxx
Communicated by the Managing Editors

Keywords:
Instanton homology
Floer homology
3-manifolds

ABSTRACT

In the description of the instanton Floer homology of a surface times a circle due to Muñoz, we compute the nilpotency degree of the endomorphism $u^2 - 64$. We then compute the framed instanton homology of a surface times a circle with non-trivial bundle, which is closely related to the kernel of $u^2 - 64$. We discuss these results in the context of the moduli space of stable rank two holomorphic bundles with fixed odd determinant over a Riemann surface.

Published by Elsevier Inc.

1. Introduction

For a closed, oriented and connected 3-manifold equipped with an $SO(3)$-bundle that restricts non-trivially to some embedded, oriented surface, Floer [5] defined a relatively $\mathbb{Z}/8$-graded complex vector space called *instanton homology*. We call such bundles *non-trivial admissible*. Floer also earlier defined his instanton homology for integral homology 3-spheres, in which case the bundle is trivial [4]. When the bundle is non-trivial, the in-

* Corresponding author.
E-mail addresses: williamb@math.bgu.ac.il (W. Chen), cscaduto@scgp.stonybrook.edu (C. Scaduto).

https://doi.org/10.1016/j.aim.2018.08.002
0001-8708/Published by Elsevier Inc.

stanton homology comes with a degree four endomorphism u, which is an isomorphism. In fact, the degree zero endomorphism $u^2 - 64$ is nilpotent, as exhibited in the work of Frøyshov [6]. Although the expression $u^2 - 64$ is in general not well-defined on Floer's instanton homology in the homology 3-sphere case, it does make sense in the framework of Frøyshov's *reduced* instanton homology groups in [6], and Frøyshov proves that u^2-64 is nilpotent in this context as well.

The instanton homology of a surface times a circle with bundle whose second Stiefel–Whitney class is Poincaré dual to the circle plays an important rôle in the general structure of instanton homology. It is equipped with a ring structure, which was computed by Muñoz [17]. Using Muñoz's ring one can also see the nilpotency of $u^2 - 64$ for non-trivial admissible bundles.

Our first result is on the degree of this nilpotency. Let Σ be a closed, oriented surface of genus $g \geqslant 1$, and denote Muñoz's instanton homology with non-trivial bundle as above by $I(\Sigma \times S^1)_w$.

Theorem 1. *The endomorphism $u^2 - 64$ acting on the instanton homology $I(\Sigma \times S^1)_w$ satisfies*

$$\min\Big\{n \geqslant 1 : \ (u^2 - 64)^n = 0\Big\} = 2\left\lceil g/2 \right\rceil - 1.$$

This in turn gives an upper bound for the nilpotency degree of $u^2 - 64$ on the instanton homology of an arbitrary 3-manifold with non-trivial admissible bundle. Following the notation of [13], suppose Y is a closed, oriented and connected 3-manifold with a Hermitian line bundle $w \longrightarrow Y$ whose first chern class has odd pairing with some closed, oriented surface $\Sigma \subset Y$. The class $w_2(w)$ determines the $SO(3)$-bundle used to define the instanton homology $I(Y)_w$. In the sequel we call the pair (Y, w) *non-trivial admissible.* In the cylinder $Y \times \mathbb{R}$, take a $\Sigma \times D^2$ neighborhood of the surface $\Sigma \times \{0\}$. By a stretching argument employed by Frøyshov, the u-map of $I(Y)_w$ factors through $I(S^1 \times \Sigma)_w$ and we have, cf. [6, Thm. 9]:

Corollary 1. *Let (Y, w) be a non-trivial admissible pair. If there is a closed oriented surface $\Sigma \subset Y$ of genus zero with $w|_\Sigma$ nontrivial, then $I(Y)_w = 0$. Otherwise, $u^2 - 64$ acting on $I(Y)_w$ satisfies*

$$\min\Big\{n \geqslant 1 : \ \ (u^2 - 64)^n = 0\Big\} \leqslant 2\left\lceil g/2 \right\rceil - 1$$

where $g \geqslant 1$ is the minimal genus over all closed oriented surfaces $\Sigma \subset Y$ with $w|_\Sigma$ non-trivial.

One may go on to deduce results regarding the instanton homology of homology 3-spheres from Theorem 1 and Floer's exact triangle, but, as we remark in the closing of this introduction, the full power of the above nilpotency degree seems most relevant for non-trivial bundles.

Download English Version:

https://daneshyari.com/en/article/8904613

Download Persian Version:

https://daneshyari.com/article/8904613

Daneshyari.com