Advances in Mathematics 335 (2018) 60–110

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Absolute continuity of non-homogeneous self-similar measures

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A R T I C L E   I N F O

Article history:
Received 1 October 2017
Received in revised form 2 May 2018
Accepted 19 June 2018
Available online xxxx
Communicated by Kenneth Falconer

MSC:
primary 28A78, 28A80
secondary 37A45, 42A38

Keywords:
Absolute continuity
Self-similar measures
Random measures
Convolutions

A B S T R A C T

We prove that self-similar measures on the real line are absolutely continuous for almost all parameters in the super-critical region, in particular confirming a conjecture of S.-M. Ngai and Y. Wang. While recently there has been much progress in understanding absolute continuity for homogeneous self-similar measures, this is the first improvement over the classical transversality method in the general (non-homogeneous) case. In the course of the proof, we establish new results on the dimension and Fourier decay of a class of random self-similar measures.

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[1] S.S. was partially supported by Iniciativa Científica Milenio NC120062 and the Israel Science Foundation research grant 1723/14.
[2] P.S. was partially supported by Project PIP 11220150100355 (CONICET). Part of this work was completed while P.S. was visiting Mittag-Leffler institute as part of the program "Fractal Geometry and Dynamics". P.S. thanks the organizers and staff for financial support and for a stimulating atmosphere.
[3] B.S. was partially supported by the Israel Science Foundation grant 396/15.

https://doi.org/10.1016/j.aim.2018.06.015

1. Introduction and main results

1.1. Non-homogeneous self-similar measures

Given $\lambda_1, \lambda_2 \in (0,1)$ and $p \in (0,1)$, one can form the self-similar measure $\nu^p_{\lambda_1,\lambda_2}$ constructed with contraction ratios λ_1, λ_2 and weight p. That is, $\nu^p_{\lambda_1,\lambda_2}$ is the only Borel probability measure ν such that

$$\nu = p\, g_{1,\lambda_1}\nu + (1-p)\, g_{2,\lambda_2}\nu,$$

where $g_{1,\lambda}(x) = \lambda x$ and $g_{2,\lambda}(x) = \lambda x + 1$. Here, and throughout the article, $g\nu$ denotes the image measure: $g\nu(A) = \nu(g^{-1}A)$ for all Borel sets A. When $\lambda_1 = \lambda_2$, the measure $\nu^p_\lambda := \nu^p_{\lambda,\lambda}$ is the classical (biased) Bernoulli convolution.

Write

$$s^p_{\lambda_1,\lambda_2} = \frac{p\log p + (1-p)\log(1-p)}{p\log\lambda_1 + (1-p)\log\lambda_2}$$

for the similarity dimension of $\nu^p_{\lambda_1,\lambda_2}$. It is well known that if $s^p_{\lambda_1,\lambda_2} < 1$, then $\nu^p_{\lambda_1,\lambda_2}$ is always singular, even though if $\lambda_1 + \lambda_2 > 1$ its support is an interval.

In the case of Bernoulli convolutions, it was recently proved by the second author [14] that there exists an exceptional set $E \subset (1/2, 1)$ of zero Hausdorff dimension, such that ν^p_λ is absolutely continuous whenever $\lambda \in (1/2, 1) \setminus E$ and $s^p_\lambda := s^p_{\lambda,\lambda} > 1$ and, moreover, ν^p_λ has a density in L^q if $\lambda^{q-1} > p^q + (1-p)^q$ (this range is sharp up to the endpoint). See Theorem 1.3, Theorem 9.1 and discussion afterwards in [14] for details. This improved several earlier results [18,12,7,13,15] providing various bounds on the size of exceptional sets for different notions of smoothness of ν^p_λ. We remark, in particular, that in an early application of the so-called "transversality method", the third author already in 1995 established absolute continuity of $\nu^{1/2}_\lambda$ for Lebesgue almost all $\lambda \in (1/2, 1)$ [18]. Also very recently, P. Varjú [20] established absolute continuity of ν^p_λ for many explicit algebraic values of λ near 1.

In the general case $\lambda_1 \neq \lambda_2$, much less is known. For $p = 1/2$, Jörg Neunhäuserer [10] and Sze-Man Ngai and Yang Wang [11] proved that $\nu^{1/2}_{\lambda_1,\lambda_2}$ is absolutely continuous for almost all λ_1, λ_2 in a certain simply connected region which is very far from covering the whole super-critical parameter region $\lambda_1\lambda_2 > 1/4$ (which corresponds to $s^{1/2}_{\lambda_1,\lambda_2} > 1$), and in particular is disjoint from a neighborhood of $(1,1)$. Ngai and Wang conjectured that, in fact, $\nu^{1/2}_{\lambda_1,\lambda_2}$ is absolutely continuous for almost all $\lambda_1, \lambda_2 \in (0,1)$ such that $\lambda_1\lambda_2 > 1/4$. This fits into the more general folklore conjecture that self-similar measures should be generically absolutely continuous in the super-critical regime. Recently, Hochman [8] proved that $\nu^p_{\lambda_1,\lambda_2}$ has Hausdorff dimension 1 for all λ_1, λ_2 such that $s^p_{\lambda_1,\lambda_2} > 1$, outside of an exceptional set of Hausdorff dimension 1, which is uniform in p. However, it does not seem possible to obtain absolute continuity from Hochman's approach.