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Nonlocal curvature and topology of locally conformally flat manifolds

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ABSTRACT

In this paper, we focus on the geometry of compact conformally flat manifolds (M^n, g) with positive scalar curvature. Schoen–Yau proved that its universal cover $(\widetilde{M^n}, \tilde{g})$ is conformally embedded in $\mathbb{S}^n$ such that M^n is a Kleinian manifold. Moreover, the limit set of the Kleinian group has Hausdorff dimension $< \frac{n-2}{2}$. If additionally we assume that the non-local curvature $Q_{2\gamma} \geq 0$ for some $1 < \gamma < 2$, the Hausdorff dimension of the limit set is less than or equal to $\frac{n-2\gamma}{2}$. If $Q_{2\gamma} > 0$, then the above inequality is strict. Moreover, the above upper bound is sharp. As applications, we obtain some topological rigidity and classification theorems in lower dimensions.

Contents

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1. Introduction

Compact locally conformally flat manifolds with positive scalar curvature can be viewed as Kleinian manifolds by Schoen–Yau's fundamental work in [17]. That is, if (M^n, g) is a compact locally conformally flat manifold with $R_g > 0$, then the universal cover $(\widetilde{M^n}, \tilde{g})$ can be conformally embedded in the standard sphere $(\mathbb{S}^n, g_1)$. Moreover, $\pi_1(M^n)$ is isomorphic to a Kleinian group $\Gamma \leq \mathrm{Conf}(\mathbb{S}^n)$ such that

$$\widetilde{M^n} \cong \Omega(\Gamma) \equiv \mathbb{S}^n \setminus \Lambda, \tag{1.1}$$

where $\Lambda \equiv \Lambda(\Gamma)$ is the limit set of the Kleinian group Γ. In [17], Schoen–Yau also proved the following Hausdorff dimension estimate on the limit set Λ in the above setting,

$$\dim_{\mathcal{H}}(\Lambda) < \frac{n-2}{2}. \tag{1.2}$$

The above Hausdorff dimension estimate immediately gives homotopy vanishing and homology vanishing results, which are interesting topological obstructions for conformally flat manifolds with nonnegative scalar curvature (see [17] for more details).

In this paper, we will generalize the above theory to the fractional setting. In conformal geometry, scalar curvature R_g arises as the zeroth order term of the conformal Laplacian operator. More precisely, denote $J_g \equiv \frac{R_g}{2(n-1)}$, then

$$P_2 \equiv -\triangle_g + \frac{n-2}{2} J_g. \tag{1.3}$$

It is standard that the second order conformal Laplacian operator P_2 satisfies the following conformal covariance property: For $n \geq 3$, let $\hat{g} = v^{\frac{4}{n-2}} g$ and let $\widehat{P}_2$ be the conformal Laplacian with respect to the conformal metric $\hat{g}$, then

$$\widehat{P}_2(u) = v^{-\frac{n+2}{n-2}} P_2(uv). \tag{1.4}$$

The fourth order analogy of P_2 is called *Paneitz operator*, which is defined by

$$P_4(u) \equiv (-\triangle_g)^2 u + \mathrm{Div}_g(4A_g\langle \nabla_g u, e_j\rangle e_j - (n-2)J_g \nabla_g u) + \frac{n-4}{2} Q_4 \cdot u, \tag{1.5}$$