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Products of Menger spaces in the Miller model

Lyubomyr Zdomskyy [1,2]

Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Straße 25, A-1090 Wien, Austria

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ABSTRACT

We prove that in the Miller model the Menger property is preserved by finite products of metrizable spaces. This answers several open questions and gives another instance of the interplay between classical forcing posets with fusion and combinatorial covering properties in topology.

1. Introduction

A topological space X has the *Menger* property (or, alternatively, is a Menger space) if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ for all n and $X = \bigcup\{\cup \mathcal{V}_n : n \in \omega\}$. This property was introduced

E-mail address: lzdomsky@gmail.com.
URL: http://www.logic.univie.ac.at/~lzdomsky/.

[1] Current address: Institute of Discrete Mathematics and Geometry, Technische Universität Wien, Wiedner Hauptstraße 8–10, 1040 Vienna, Austria.

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by Hurewicz, and the current name (the Menger property) is used because Hurewicz proved in [15] that for metrizable spaces his property is equivalent to one considered by Menger in [19]. Each σ-compact space has obviously the Menger property, and the latter implies lindelöfness (that is, every open cover has a countable subcover). The Menger property is the weakest one among the so-called *selection principles* or *combinatorial covering properties*, see, e.g., [4,17,25,26,32] for detailed introductions to the topic. Menger spaces have recently found applications in such areas as forcing [12], Ramsey theory in algebra [34], combinatorics of discrete subspaces [2], and Tukey relations between hyperspaces of compacts [14].

In this paper we proceed our investigation of the interplay between posets with fusion and selection principles initiated in [24]. More precisely, we concentrate on the question whether the Menger property is preserved by finite products. For general topological spaces the answer negative: In ZFC there are two normal spaces X, Y with a covering property much stronger than the Menger one such that $X \times Y$ does not have the Lindelöf property, see [29, §3]. However, the above situation becomes impossible if we restrict our attention to metrizable spaces. This case, on which we concentrate in the sequel, turned out to be sensitive to the ambient set-theoretic universe. Indeed, by [22, Theorem 3.2] under CH there exist $X, Y \subset \mathbb{R}$ which have the Menger property (in fact, they have the strongest combinatorial covering property considered thus far), whose product $X \times Y$ is not Menger. There are many results of this kind where CH is relaxed to an equality between cardinal characteristics, see, e.g., [3,16,23,27]. Surprisingly, there are also inequalities between cardinal characteristics which imply that the Menger property is not productive even for sets of reals, see [28]. The following theorem, which is the main result of our paper, shows that an additional set-theoretic assumption in all these results was indeed necessary.

Theorem 1.1. *In the Miller model, the product of any two Menger spaces is Menger provided that it is Lindelöf. In particular, in this model the product of any two Menger metrizable spaces is Menger.*

Theorem 1.1 answers [28, Problem 7.9(2)], [30, Problem 8.4] (restated as [31, Problem 4.11]), and [33, Problem 6.7] in the affirmative; implies that the affirmative answer to [1, Problem II.2.8], [5, Problem 3.9], and [16, Problem 2] (restated as [32, Problem 3.2] and [33, Problem 2.1]) is consistent; implies that the negative answer to and [1, Problem II.2.7], [6, Problem 8.9], and [36, Problems 1, 2, 3] is consistent; and answers [25, Problem 7] in the negative.

By the *Miller model* we mean a generic extension of a ground model of GCH with respect to the iteration of length ω_2 with countable support of the Miller forcing, see the next section for its definition. This model has been first considered by Miller in [20] and since then found numerous applications, see [7] and references therein. The Miller forcing is similar to the Laver one introduced in [18], the main difference being that the splitting is allowed to occur less often. The main technical part of the proof of Theorem 1.1 is