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The combinatorial invariance conjecture for parabolic Kazhdan–Lusztig polynomials of lower intervals

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The aim of this work is to prove a conjecture related to the Combinatorial Invariance Conjecture of Kazhdan–Lusztig polynomials, in the parabolic setting, for lower intervals in every arbitrary Coxeter group. This result improves and generalizes, among other results, the main results of Brenti et al. (2006) [6], Marietti (2016) [21].

1. Introduction

Kazhdan–Lusztig polynomials play a central role in Lie theory and representation theory. They are polynomials $P_{u,v}(q)$, in one variable q, which are associated to pairs of elements u, v in a Coxeter group W. They were defined by Kazhdan and Lusztig in [19]

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in order to introduce the (now called) Kazhdan–Lusztig representations of the Hecke algebra of W, and soon have found applications in many other contexts.

Among others, the combinatorial aspects of Kazhdan–Lusztig polynomials have received much attention from the start, and are still a fascinating field of research. Recently, Elias and Williamson [15] proved the long-standing conjecture about the nonnegativity of the coefficients of Kazhdan–Lusztig polynomials of all Coxeter groups, thus generalizing the analogous result by Kazhdan and Lusztig on finite and affine Weyl groups appearing in [20], where $P_{u,v}(q)$ is shown to be the Poincaré polynomial of the local intersection cohomology groups of the Schubert variety associated with v at any point of the Schubert variety associated with u (in the full flag variety).

At present, from a combinatorial point of view, the most intriguing conjecture about Kazhdan–Lusztig polynomials is arguably what is usually referred to as the Combinatorial Invariance Conjecture of Kazhdan–Lusztig polynomials. It was independently formulated by Lusztig in private and by Dyer in [13].

Conjecture 1.1. *The Kazhdan–Lusztig polynomial $P_{u,v}(q)$ depends only on the isomorphism class of the interval $[u, v]$ as a poset.*

The Combinatorial Invariance Conjecture of Kazhdan–Lusztig polynomials is equivalent to the analogous conjecture on the combinatorial invariance of Kazhdan–Lusztig R-polynomials. These also are polynomials $R_{u,v}(q)$ indexed by a pair of elements u, v in W and were introduced by Kazhdan–Lusztig in the same article [19]. The Kazhdan–Lusztig R-polynomials are equivalent to the Kazhdan–Lusztig polynomials of W (in a precise sense, see Remark 2.12).

In [11], for any choice of a subset $H \subseteq S$, Deodhar introduces two modules of the Hecke algebra of W, two parabolic analogues $\{P^{H,x}_{u,w}(q)\}_{u,w\in W^H}$ of the Kazhdan–Lusztig polynomials, and two parabolic analogues $\{R^{H,x}_{u,w}(q)\}_{u,w\in W^H}$ of the Kazhdan–Lusztig R-polynomials, one for $x = q$ and one for $x = -1$. The parabolic Kazhdan–Lusztig and R-polynomials have deep algebraic and geometric significance; they are indexed by pairs of elements in the set W^H of minimal coset representatives with respect to the standard parabolic subgroup W_H, and play, in the parabolic setting, a role that is parallel to the role that the ordinary Kazhdan–Lusztig and R-polynomials play in the ordinary setting. Moreover, they generalize the ordinary Kazhdan–Lusztig and R-polynomials since these are obtained in the special trivial case when $H = \emptyset$ (for both $x = q$ and -1). As in the ordinary case, the family of parabolic Kazhdan–Lusztig polynomials is equivalent to the family of parabolic Kazhdan–Lusztig R-polynomials.

The problem of the combinatorial invariance of parabolic Kazhdan–Lusztig polynomials, which is stronger than the combinatorial invariance of the ordinary Kazhdan–Lusztig polynomials, has also attracted much attention (see, for instance, [2] and [4]). Only recently, however, the statement one gets by replacing the ordinary interval with the parabolic interval in Conjecture 1.1 has been found to be false (see [7] and [21] for counterexamples). In [21], it is proposed that the right approach to the generalization of