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Continuity of depinning force

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ABSTRACT

Monotone recurrence relations give rise to a class of dynamical systems on the high-dimensional cylinder which generalizes the class of monotone twist maps. A solution of a monotone recurrence relation corresponds to an equilibrium of the generalized Frenkel–Kontorova (FK) model. The Aubry–Mather theory for monotone recurrence relations says that for each $\omega \in \mathbb{R}$ there is a Birkhoff minimizer with rotation number ω. Whether all Birkhoff minimizers of rotation number ω form a foliation is a question like whether there is an invariant circle with rotation number ω for a monotone twist map. In this paper we give a criterion for the existence of minimal foliations and study its continuity.

The depinning force $F_d(\omega)$, depending on rotation numbers, is a critical value of the driving force for the FK model, under which there are Birkhoff equilibria and hence the system is pinned, and above which there are no Birkhoff equilibria of rotation number ω and the system is sliding. We show that all Birkhoff minimizers with irrational rotation number ω form a foliation if and only if $F_d(\omega) = 0$. If $\omega = p/q$ is rational, then $F_d(p/q) = 0$ if and only if all (p, q)-periodic Birkhoff minimizers constitute a foliation. Moreover, we show that $F_d(\omega)$ is continuous at irrational ω and Hölder continuous

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at Diophantine points, and it also depends continuously on system parameters.

1. Introduction

One of the important topics for dynamics of area-preserving monotone twist maps is to establish criteria to determine whether there is an invariant circle of rotation number ω. Mather [23,24,26] has shown that the existence of the invariant circle with rotation number ω is equivalent to the Peierls barrier $P_\omega(\xi) \equiv 0$. The modulus of continuity for the Peierls barrier has also been studied by Mather in [24]. Such conclusions are crucial to study the destruction of invariant circles [25,26]. Recently, Mramor and Rink [31,33] improved and extended these results to monotone recurrence relations.

Let $1 \leq r \in \mathbb{N}$ be a natural number indicating the range of interactions between particles. Let $h \in C^2(\mathbb{R}^{r+1}, \mathbb{R})$ satisfy some hypotheses (H1)–(H4) specified in Section 2. For each configuration $\mathbf{x} = (x_i) \in \mathbb{R}^{\mathbb{Z}}$, define the formal Lagrangian $W(\mathbf{x}) = \sum_{i\in\mathbb{Z}} h_i(\mathbf{x})$, where $h_i(\mathbf{x}) = h(x_i, \cdots, x_{i+r})$, $i \in \mathbb{Z}$.

A monotone recurrence relation is defined by finding a stationary point $\mathbf{x}$ of W:

$$\partial_i W(\mathbf{x}) = \sum_{j=i-r}^{i} \partial_i h_j(\mathbf{x}) = 0, \quad i \in \mathbb{Z}. \tag{1.1}$$

We call h the generating function of the monotone recurrence relation (1.1). Indeed, for $r = 1$, h is the generating function of a monotone twist map, see [6] or Section 2 in [1]. From the point of view of physical applications, the generating function h describes the interaction of a particle with its neighborhood and the Lagrangian W denotes the energy of a system of particles.

By the twist condition (H4) of h we can define a diffeomorphism from $\mathbb{R}^{2r}$ to $\mathbb{R}^{2r}$ and furthermore by the periodicity condition (H1) a diffeomorphism on the $2r$-dimensional cylinder, which generalizes monotone twist maps, see Section 10 in [1] or [16,39].

The Aubry–Mather theory for monotone recurrence relations has been extensively studied [7,12,13,20,30]: For each $\omega \in \mathbb{R}$, there exist Birkhoff solutions of (1.1), which are global minimizers of W. If we denote by $\mathscr{M}_\omega$ the set of all Birkhoff minimizers with rotation number ω, then $\mathscr{M}_\omega \neq \emptyset$ and it is totally ordered if ω is irrational [32].

According to Moser [28,29], $\mathscr{M}_\omega$ is called a foliation if $p_0(\mathscr{M}_\omega) = \mathbb{R}$, where p_0 is the projection on 0-site, i.e., $p_0(\mathbf{x}) = x_0$. Otherwise it is called a lamination. For monotone twist maps, foliations correspond to rotational invariant circles.

Note that rotational invariant circles for monotone twist maps are important because they form the energy transport barrier for the map. For solid state physics models, minimal foliations are associated to sliding effect and laminations to a pinning of particles.