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Deformations of the braid arrangement and trees ☆

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ABSTRACT

We establish general counting formulas and bijections for deformations of the braid arrangement. Precisely, we consider real hyperplane arrangements such that all the hyperplanes are of the form $x_i - x_j = s$ for some integer s. Classical examples include the braid, Catalan, Shi, semiorder and Linial arrangements, as well as graphical arrangements. We express the number of regions of any such arrangement as a signed count of decorated plane trees. The characteristic and coboundary polynomials of these arrangements also have simple expressions in terms of these trees.
We then focus on certain "well-behaved" deformations of the braid arrangement that we call *transitive*. This includes the Catalan, Shi, semiorder and Linial arrangements, as well as many other arrangements appearing in the literature. For any transitive deformation of the braid arrangement we establish a simple bijection between regions of the arrangement and a set of labeled plane trees defined by local conditions. This answers a question of Gessel.

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1. Introduction

In this article we establish enumerative and bijective results about classical families of hyperplane arrangements. Specifically, we consider real hyperplane arrangements made of a finite number of hyperplanes of the form

$$H_{i,j,s} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i - x_j = s\},$$

with $i, j \in \{1, \dots, n\}$ and $s \in \mathbb{Z}$. We shall call them *deformations of the braid arrangement*. In particular, given an integer n and a finite set of integers S, the *S-braid arrangement in dimension n*, denoted $\mathcal{A}_S(n)$, is the arrangement made of the hyperplanes $H_{i,j,s}$ for all $1 \leq i < j \leq n$ and all $s \in S$. Classical examples include the *braid*, *Catalan*, *Shi*, *semiorder*, and *Linial* arrangements, which correspond to $S = \{0\}$, $\{-1, 0, 1\}$, $\{0, 1\}$, $\{-1, 1\}$, and $\{1\}$ respectively. These arrangements are represented[1] in Fig. 1. We refer the reader to [34] or [44] for an introduction to the general theory of hyperplane arrangements.

There is an extensive literature on counting regions of deformations of the braid arrangement, starting with the work of Shi [38,39]. Important seminal results on this enumerative question were established by Stanley [41,42], Postnikov and Stanley [37], and Athanasiadis [5,6]. Since then, the subject has become quite popular among combinatorialists, and many beautiful counting formulas and bijections were discovered for various families of arrangements; see in particular [7,10,11,1,3,15,23,19,25,27,26,31] (see also Section 2 for additional references).

About a decade ago, an interesting pattern was observed by Ira Gessel. In an unpublished manuscript, Gessel obtained an equation for the generating function of labeled binary trees counted according to ascents and descents along left or right edges.[2] By specializing his equation, Gessel observed (based on known results for arrangements) that each of the five classical arrangements families $\mathcal{A}_S$ defined above (braid, Catalan, Shi, semi-order, Linial) can be associated to a simple family $\mathcal{B}_S$ of binary trees (charac-

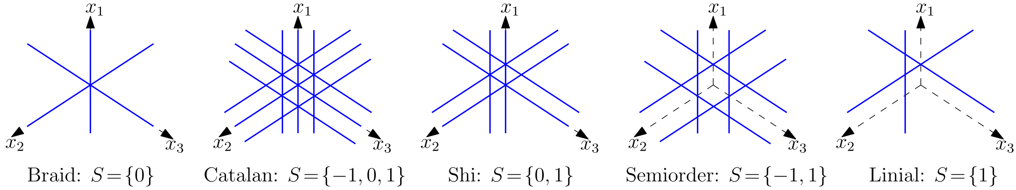

Fig. 1. The braid, Catalan, Shi, semiorder, and Linial arrangements in dimension $n = 3$ (seen from the direction $(1, 1, 1)$).

[1] Here and later, the arrangements are represented by drawing their intersection with the hyperplane $H_0 = \{(x_1, \dots, x_n) \mid x_1 + \dots + x_n = 0\}$, which is orthogonal to all the hyperplanes of any deformation of the braid arrangement.

[2] Gessel's result was then rederived in two different ways by Kalikow [28] and Drake [17].