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Primary operations in differential cohomology

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ABSTRACT

We characterize primary operations in differential cohomology via stacks, and illustrate by differentially refining Steenrod squares and Steenrod powers explicitly. This requires a delicate interplay between integral, rational, and mod p cohomology, as well as cohomology with $U(1)$ coefficients and differential forms. Along the way we develop computational techniques in differential cohomology, including a Künneth decomposition, that should also be useful in their own right, and point to applications to higher geometry and mathematical physics.

Contents

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1. Introduction

Cohomology operations with coefficients in a group G are natural transformations of the form $H^n(-;G) \to H^m(-;G)$. By Brown representability and the Yoneda lemma this is equivalent to calculating the universal cohomology group of Eilenberg–MacLane spaces $H^m(K(G,n);G)$, which is in turn equivalent to calculating the homotopy classes of maps $[K(G,n), K(G,m)]$. Thus all cohomology operations of fixed degree are accounted for by calculating the cohomology of the Eilenberg–MacLane space $K(G,n)$. To account for all cohomology operations one obviously has to vary both m and n. See [37] [46] for detailed accounts.

We are interested in differential cohomology (see [10] [19] [28] [45] [5] [6] [44] [1]). What replaces Eilenberg–MacLane spaces are various stacks of higher $U(1)$-bundles (n-bundles) with connections. Thus, cohomology operations will involve the differential cohomology of such stacks, and this process can be described via mapping spaces of stacks. For differential refinements we will need to study morphisms of stacks

$$\widehat{\theta} : \mathbf{B}^n U(1)_\nabla \longrightarrow \mathbf{B}^m U(1)_\nabla \ , \tag{1.1}$$

where $\mathbf{B}^n U(1)_\nabla$ represents the moduli stack of n-bundles equipped with connection, studied in [13][15][16]. The homotopy classes of such morphisms will in turn describe the differential cohomology group

$$\widehat{H}^{k+1}(\mathbf{B}^n U(1)_\nabla; \mathbb{Z}) := \pi_0 \mathrm{Map}\big(\mathbf{B}^n U(1)_\nabla, \mathbf{B}^k U(1)_\nabla\big) \ .$$

One of the main goals of this paper is to characterize this group for various values of k and n. This in turn will lead to a full characterization of primary cohomology operations in differential cohomology.

Since differential cohomology operations, as we will see, involve various coefficients, we find it useful to point out the interrelations that already exist between these (we found the discussion in [18] particularly useful). This should also help us develop some intuition for the full differential case. Note that for coefficients being one of $\mathbb{Z}, \mathbb{Z}/p$ or $\mathbb{Q}$ i.e. an abelian group, then the set of all cohomology operations $H^m(K(G,n);G)$, where G and G' are from the above set, will also be abelian.

Operations from $\mathbb{Z}/p$ to $\mathbb{Q}$. We know that $H^q(K(G,n);\mathbb{Q}) = 0$ for all $q > 0$ when G is a finite abelian group, i.e. for us $\mathbb{Z}/p$. This shows that there are no nontrivial cohomology operations from $\mathbb{Z}/p$-coefficients to $\mathbb{Q}$-coefficients.