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# Positivity of the diagonal

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A B S T R A C T

We study how the geometry of a projective variety $X$ is reflected in the positivity properties of the diagonal $\Delta_X$ considered as a cycle on $X \times X$. We analyze when the diagonal is big, when it is nef, and when it is rigid. In each case, we give several implications for the geometric properties of $X$. For example, when the cohomology class of $\Delta_X$ is big, we prove that the Hodge groups $H^{k,0}(X)$ vanish for $k > 0$. We also classify varieties of low dimension where the diagonal is nef and big.



## 1. Introduction

The geometry of a projective variety $X$ is determined by the positivity of the tangent bundle $T_X$. Motivated by the fact that $T_X$ is the normal bundle of the diagonal $\Delta_X$ in the self-product $X \times X$, we will in this paper study how the geometry of $X$ is reflected in the positivity properties of $\Delta_X$ itself, considered as a cycle on $X \times X$. The prototypical

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example of a variety with positive diagonal is projective space; the central theme of the paper is that positivity of the diagonal forces $X$ to be similar to projective space. In dimension 1, this perspective is already quite vivid: $\mathbb{P}^1$ is the only curve where the diagonal is an ample divisor; elliptic curves have nef, but not big diagonals; and for higher genus, the diagonal is contractible, hence 'negative' in a very strong sense.

In general, when $X$ has dimension $n$, the diagonal determines a class in the space $N_n(X \times X)$ of $n$-dimensional cycles modulo numerical equivalence, and we are interested in how this class sits with respect to the various cones of positive cycles of $X \times X$. Note that in the absence of the Hodge conjecture, we often do not even know the dimension of the space $N_n(X \times X)$. Thus we develop techniques to prove positivity or rigidity without an explicit calculation of the positive cones.

The subsections below recall several different types of positivity and give a number of theorems illustrating each. At the end of the introduction we will collect several examples of particular interest.

### 1.1. Big diagonal

A cycle class $\alpha$ is said to be *big* if it lies in the interior of the closed cone generated by classes of effective cycles. Bigness is perhaps the most natural notion of positivity for cycles. We will also call a cycle *homologically big* if it is homologically equivalent to the sum of an effective $\mathbb{Q}$-cycle and a complete intersection of ample $\mathbb{Q}$-divisors. Homological bigness implies bigness, and equivalence of the two notions would follow from the standard conjectures.

The primary example of a variety with (homologically) big diagonal is projective space. In this case, the diagonal has a Künneth decomposition of the form

$$\Delta_X = \sum_{p+q=n} \pi_1^* h^p \cdot \pi_2^* h^q$$

where $h$ is the hyperplane divisor, and this class is evidently big. Of course, the same argument applies also for the *fake projective spaces*, that is, smooth varieties $\neq \mathbb{P}^n$ with the same betti numbers as $\mathbb{P}^n$. In dimension 2, there are exactly 100 such surfaces [31], [7], and they are all of general type. Thus unlike the case of curves, we now allow examples with positive Kodaira dimension, but which are still 'similar' to projective space in the sense that they have the same Hodge diamond.

More generally, homological bigness of the diagonal implies the vanishing of the 'outer' Hodge groups of $X$. Following ideas of [13], we show:

**Theorem 1.1.** *Let $X$ be a smooth projective variety. If $\Delta_X$ is homologically big, then $H^{i,0}(X) = 0$ for $i > 0$.*

An interesting feature of this result is that the proof makes use of *non-algebraic* cohomology classes to control effective cycles. When $X$ is a surface with a big diagonal,