Advances in Mathematics 335 (2018) 809–837

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Triply periodic constant mean curvature surfaces

William H. Meeks III [a,1], Giuseppe Tinaglia [b,*,2]

[a] *Mathematics Department, University of Massachusetts, Amherst, MA 01003, United States*

[b] *Department of Mathematics, King's College London, London, WC2R 2LS, UK*

ARTICLE INFO

Article history:
Received 27 May 2017
Received in revised form 8 June 2018
Accepted 2 July 2018
Available online 26 July 2018
Communicated by Gang Tian

MSC:
primary 53A10
secondary 49Q05, 53C42

Keywords:
Minimal surface
Constant mean curvature
H-lamination
Injectivity radius
Curvature estimates
Area estimates

ABSTRACT

Given a closed flat 3-torus N, for each $H > 0$ and each non-negative integer g, we obtain area estimates for closed surfaces with genus g and constant mean curvature H embedded in N. This result contrasts with the theorem of Traizet [31], who proved that every flat 3-torus admits for every positive integer g with $g \neq 2$, connected closed embedded minimal surfaces of genus g with arbitrarily large area.

* Corresponding author.
E-mail addresses: profmeeks@gmail.com (W.H. Meeks), giuseppe.tinaglia@kcl.ac.uk (G. Tinaglia).

[1] The first author was supported in part by NSF Grant DMS-1309236. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

[2] The second author was partially supported by EPSRC grant no. EP/M024512/1.

https://doi.org/10.1016/j.aim.2018.07.018
0001-8708/

1. Introduction

In [31] Traizet proved that every flat 3-torus admits for every positive integer g with $g \neq 2$, connected closed embedded minimal surfaces of genus g with arbitrarily large area. In contrast to Traizet's result we prove the following area estimates for closed embedded surfaces of constant positive mean curvature in a flat 3-torus.

Theorem 1.1. *Given $a, b, d, I_0 \in (0, \infty)$ with $a \leq b$ and $g \in \mathbb{N} \cup \{0\}$, there exists $A(g, a, b, d, I_0) > 0$ such that the following hold. Let N be a flat 3-torus with an upper bound d on its diameter and a lower bound I_0 for its injectivity radius and let M be a possibly disconnected, closed surface embedded in N of genus g with constant mean curvature $H \in [a, b]$. Then:*

$$\text{Area}(M) \leq A(g, a, b, d, I_0).$$

Interest in results like the above area estimates arises in part from the fact that triply periodic surfaces of constant mean curvature in $\mathbb{R}^3$, which are always the lifts of surfaces of constant mean curvature in a flat 3-torus, occur in nature. For example, approximations to these special surfaces appear in studies of Fermi surfaces (equipotential surfaces) in solid state physics and in the geometry of liquid crystals. They are also found in material sciences where they closely approximate surface interfaces in certain inhomogeneous mixtures of two different compounds and as the boundary of the microscopic calcium deposit patterns in sea urchin shells. The geometry of triply-periodic constant mean curvature surfaces arising in nature have profound consequences for the physical properties of the materials in which they occur, which is one of the reasons why they are studied in great detail by physical scientists. We refer the interested reader to the science articles [1,9,28] for further readings along these lines.

For the purpose of exposition, in this paper we will call an orientable immersed surface M in an oriented Riemannian 3-manifold an *H-surface* if M is *embedded* and it has *non-zero constant mean curvature H*, where we will assume that H is positive by appropriately orienting M.

A consequence of Theorem 1.1 and its proof is that for $H > 0$ fixed, the moduli space of closed connected H-surfaces of fixed genus in a flat 3-torus has a natural compactification as a compact real semi-analytic variety in much the same way that one can compactify the moduli space of closed connected Riemann surfaces of fixed genus. In this regard it is worthwhile to compare our results with the area estimates of Choi and Wang [4] for closed embedded minimal surfaces of fixed genus in the 3-sphere $\langle \mathbb{S}^3, h \rangle$ with a metric h of positive Ricci curvature, and with the result by Choi and Schoen [3] that the moduli space of fixed genus closed minimal surfaces embedded in $\langle \mathbb{S}^3, h \rangle$ has the structure of a compact real analytic variety.

Theorem 1.2. *Let N be a flat 3-torus. Given $a, b \in (0, \infty)$ with $a \leq b$ and $g \in \mathbb{N} \cup \{0\}$, let $\{M_n\}_{n \in \mathbb{N}}$ be a sequence of closed H_n-surfaces in N, $H_n \in [a, b]$, of genus g. Then there*