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A uniform realization of the combinatorial R-matrix for column shape Kirillov–Reshetikhin crystals ☆

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ABSTRACT

Kirillov–Reshetikhin (KR) crystals are colored directed graphs encoding the structure of certain finite-dimensional representations of affine Lie algebras. A tensor product of column shape Kirillov–Reshetikhin crystals has recently been realized in a uniform way, for all untwisted affine types, in terms of the so-called quantum alcove model. We enhance this model by using it to give a uniform realization of the corresponding combinatorial R-matrix, i.e., the unique affine crystal isomorphism permuting factors in a tensor product of column shape KR crystals. In other words, we are generalizing to all Lie types Schützenberger's sliding game (jeu de taquin) for Young tableaux, which realizes the combinatorial R-matrix in type A. Our construction is in terms of certain combinatorial moves, called quantum Yang–Baxter moves, which are explicitly described by reduction to the rank 2 root systems. We also show that the quantum alcove model does not depend on the choice of a sequence of alcoves joining the fundamental one to a translation of it.

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1. Introduction

Kashiwara's *crystals* [14] are colored directed graphs encoding the structure of certain bases (called crystal bases) of some representations of quantum groups $U_q(\mathfrak{g})$ as q goes to zero (where $\mathfrak{g}$ is a symmetrizable Kac–Moody Lie algebra). All highest weight integrable representations have crystal bases/graphs. Beside them, an important class of crystals is represented by the *Kirillov–Reshetikhin (KR) crystals* [16]. They correspond to certain finite-dimensional modules for affine Lie algebras which are not of highest weight. A KR crystal is denoted $B^{r,s}$, being labeled by the multiple $s\omega_r$ of the fundamental weight ω_r, where s is any positive integer. When $s = 1$ we refer to the corresponding KR crystals as having column shape; this terminology is justified by the fact that, in classical types, there are reasons to think of the pair (r, s) as a rectangle with height r and width s. The importance of KR crystals stems from the fact that they are building blocks for the corresponding (infinite) highest weight crystals; indeed, the latter are realized as infinite tensor products of the former in the Kyoto path model, see, e.g., [13]. Tensor products of KR crystals are endowed with a grading known as the *energy function* [32,34], which originates in the theory of solvable lattice models [12]. There is a unique affine crystal isomorphism between two tensor products of KR crystals differing by a permutation of the tensor factors; it is called the *combinatorial R-matrix*.

The first author and Postnikov [27,28] defined the so-called *alcove model* for highest weight crystals associated to a symmetrizable Kac–Moody algebra. A related model is the one of Gaussent–Littelmann, based on LS-galleries [11]. Both models are discrete counterparts of the celebrated *Littelmann path model* [30,31]. In [22] the authors generalize the alcove model. The main ingredient is the so-called *quantum Bruhat graph* structure on the Weyl group, which replaces the (strong) Bruhat order in the case of the alcove model. This generalization, thus called the *quantum alcove model*, has been shown in [25] to uniformly describe tensor products of column shape KR crystals for all untwisted affine types; cf. also [26]. By contrast, all the existing combinatorial models for KR crystals are type-specific; most of them correspond to the classical types, and are based on diagram fillings, i.e., on tableau models [7]. The quantum alcove model also has the advantage of allowing a uniform and efficient calculation of the energy function as a statistic called height [25].

In this paper we enhance the quantum alcove model by using it to give a uniform realization of the combinatorial R-matrix for tensor products of column shape KR crystals. The construction is based on certain combinatorial moves called *quantum Yang–Baxter moves*, which generalize their alcove model versions defined in [20] and called *Yang–Baxter moves*; the terminology is justified by the connection to the Yang–Baxter equation, see [27, Section 9] and [3]. The quantum Yang–Baxter moves are explicitly described in all Lie types by reduction to rank 2 root systems. We exhibit several properties of these moves, such as their commutation with the crystal operators, and the preservation of the height statistic (which computes the energy) and of the weight; some properties (such as the one related to the height statistic) are intrinsic to the quantum