Advances in Mathematics 334 (2018) 243–299

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

The nodal set of solutions to some elliptic problems: Sublinear equations, and unstable two-phase membrane problem ☆

Nicola Soave [a,*], Susanna Terracini [b]

[a] *Dipartimento di Matematica, Politecnico di Milano, Via Edoardo Bonardi 9, 20133 Milano, Italy*
[b] *Dipartimento di Matematica "Giuseppe Peano", Università di Torino, Via Carlo Alberto, 10, 10123 Torino, Italy*

A R T I C L E   I N F O

Article history:
Received 24 April 2018
Received in revised form 13 June 2018
Accepted 19 June 2018
Available online xxxx
Communicated by O. Savin

MSC:
35B05
35R35
35J60
35B60
28A78

Keywords:
Sublinear equations
Unstable two-phase obstacle problem
Nodal set
Nondegeneracy
Stratification

A B S T R A C T

We are concerned with the nodal set of solutions to equations of the form

$$-\Delta u = \lambda_+ \left(u^+\right)^{q-1} - \lambda_- \left(u^-\right)^{q-1} \quad \text{in } B_1$$

where $\lambda_+, \lambda_- > 0$, $q \in [1,2)$, $B_1 = B_1(0)$ is the unit ball in $\mathbb{R}^N$, $N \geq 2$, and $u^+ := \max\{u, 0\}$, $u^- := \max\{-u, 0\}$ are the positive and the negative part of u, respectively. This class includes, the *unstable two-phase membrane problem* ($q = 1$), as well as *sublinear* equations for $1 < q < 2$.

We prove the following main results: (a) the finiteness of the vanishing order at every point and the complete characterization of the order spectrum; (b) a weak non-degeneracy property; (c) regularity of the nodal set of any solution: the nodal set is a locally finite collection of regular codimension one manifolds up to a residual singular set having Hausdorff dimension at most $N - 2$ (locally finite when $N = 2$); (d) a partial stratification theorem.

☆ The authors are partially supported by the ERC Advanced Grant 2013 No. 339958 "Complex Patterns for Strongly Interacting Dynamical Systems – COMPAT". N. Soave is partially supported by the PRIN-2015KB9WPT_010 Grant: "Variational methods, with applications to problems in mathematical physics and geometry", and by the INDAM-GNAMPA project "Aspetti non-locali in fenomeni di segregazione". Declarations of interest: none.

* Corresponding author.
E-mail addresses: nicola.soave@gmail.com, nicola.soave@polimi.it (N. Soave), susanna.terracini@unito.it (S. Terracini).

https://doi.org/10.1016/j.aim.2018.06.007
0001-8708/

Ultimately, the main features of the nodal set are strictly related with those of the solutions to linear (or superlinear) equations, with two remarkable differences. First of all, the admissible vanishing orders can not exceed the critical value $2/(2-q)$. At threshold, we find a multiplicity of homogeneous solutions, yielding the *non-validity* of any estimate of the $(N-1)$-dimensional measure of the nodal set of a solution in terms of the vanishing order.
As a byproduct, we also prove the strong unique continuation property for the *unstable obstacle problem*, corresponding to the case $\lambda_- = 0$.
The proofs are based on monotonicity formulæ for a 2-parameter family of Weiss-type functionals, blow-up arguments, and the classification of homogeneous solutions.

1. Introduction

In this paper we investigate the structure of the nodal set of solutions to

$$-\Delta u = \lambda_+ \left(u^+\right)^{q-1} - \lambda_- \left(u^-\right)^{q-1} \quad \text{in } B_1 \tag{1.1}$$

where $\lambda_+ > 0$, $\lambda_- \geq 0$, $q \in [1,2)$, $B_1 = B_1(0)$ is the unit ball in $\mathbb{R}^N$, $N \geq 2$, and $u^+ := \max\{u, 0\}$, $u^- := \max\{-u, 0\}$ are the positive and the negative part of u, respectively. The main feature of the problem stays in the fact that the right hand side is not locally Lipschitz continuous as function of u, and precisely has sublinear character for $1 < q < 2$, and discontinuous character for $q = 1$. Our study is driven by two main motivations: the comparison with the structure of the nodal set of solutions to linear problems and the investigation of the free boundaries of unstable obstacle problems with two phases.

The study of the nodal set of solutions to second order linear (or superlinear) elliptic equations stimulated a very intense research, starting from the seminal contribution by T. Carleman regarding the validity of the strong unique continuation principle [10]. Many generalizations of Carleman's result are now available, we refer the reader to [27] and the references therein for a more detailed discussion. In a slightly different direction, researcher also analyzed the structure of the nodal sets from the geometric point of view. For a weak solution v of class at least C^1 (this is often the case by regularity theory), the nodal set splits into a regular part where $\nabla v \neq 0$ and a singular (or critical) set where v vanishes together with its gradient. The regular part is in fact locally a C^1 graph by the implicit function theorem, so that the study of the nodal set reduces to the study of its singular subset. The first results concerning the structure of the singular set are due to L. Caffarelli and A. Friedman [8,9], who proved the partial regularity of the nodal set of solutions to semilinear elliptic equations driven by the Laplacian and with linear or superlinear right hand side; that is, their singular set has Hausdorff dimension at