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A rigidity result for effective Hamiltonians with 3-mode periodic potentials ☆

Hung V. Tran [a], Yifeng Yu [b,*]

[a] Department of Mathematics, University of Wisconsin Madison, Van Vleck hall, 480 Lincoln drive, Madison, WI 53706, USA

[b] Department of Mathematics, University of California, Irvine, 410G Rowland Hall, Irvine, CA 92697, USA

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ABSTRACT

We continue studying an inverse problem in the theory of periodic homogenization of Hamilton–Jacobi equations proposed in [14]. Let $V_1, V_2 \in C(\mathbb{R}^n)$ be two given potentials which are $\mathbb{Z}^n$-periodic, and $\overline{H}_1, \overline{H}_2$ be the effective Hamiltonians associated with the Hamiltonians $\frac{1}{2}|p|^2+V_1$, $\frac{1}{2}|p|^2+V_2$, respectively. A main result in this paper is that, if the dimension $n=2$, and each of V_1, V_2 contains exactly 3 mutually non-parallel Fourier modes, then

$$\overline{H}_1 \equiv \overline{H}_2 \iff V_1(x) = V_2\left(\frac{x}{c}+x_0\right) \quad \text{for all } x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2,$$

for some $c \in \mathbb{Q}\setminus\{0\}$ and $x_0 \in \mathbb{T}^2$. When $n \geq 3$, the scenario is slightly more subtle, and a complete description is provided for any dimension. These resolve partially a conjecture stated in [14]. Some other related results and open problems are also discussed.

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* Corresponding author.
E-mail addresses: hung@math.wisc.edu (H.V. Tran), yyu1@math.uci.edu (Y. Yu).

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1. Introduction

1.1. Periodic homogenization and the inverse problem

We first describe the theory of periodic homogenization of Hamilton–Jacobi equations. For each length scale $\varepsilon > 0$, let $u^\varepsilon \in C(\mathbb{R}^n \times [0,\infty))$ be the viscosity solution to

$$\begin{cases} u_t^\varepsilon + H(Du^\varepsilon) + V\left(\frac{x}{\varepsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here, the Hamiltonian $H(p) - V(x)$ is of separable form with $H \in C(\mathbb{R}^n)$, which is coercive (i.e., $\lim_{|p|\to\infty} H(p) = +\infty$), and $V \in C(\mathbb{R}^n)$, which is $\mathbb{Z}^n$-periodic. The initial data $g \in \mathrm{BUC}(\mathbb{R}^n)$, the set of bounded, uniformly continuous functions on $\mathbb{R}^n$.

It was shown in [13] that, in the limit as the length scale ε tends to zero, u^ε converges to u locally uniformly on $\mathbb{R}^n \times [0,\infty)$, and u solves the effective equation

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.2}$$

The effective Hamiltonian $\overline{H} \in C(\mathbb{R}^n)$ is determined in a nonlinear way by H and V through the cell problems as follows. For each $p \in \mathbb{R}^n$, it was derived in [13] that there exists a unique constant $c \in \mathbb{R}$ such that the following cell problem has a continuous viscosity solution

$$H(p + Dv) + V(x) = c \quad \text{in } \mathbb{T}^n, \tag{1.3}$$

where $\mathbb{T}^n$ is the n-dimensional flat torus $\mathbb{R}^n/\mathbb{Z}^n$. We then denote by $\overline{H}(p) := c$.

During past decades, there have been tremendous progress and vast literature about the validity of homogenization and the well-posedness of cell problems in various generalized settings. Nevertheless, understanding theoretically how $\overline{H}$ depends on the potential V remains a very challenging and still largely open problem even for the most basic case $H(p) = \frac{1}{2}|p|^2$. For a smooth periodic potential V, a deep result in [4] asserts that when $n = 2$ and $H(p) = \frac{1}{2}|p|^2$, each non-minimum level curve of $\overline{H}$ associated with $\frac{1}{2}|p|^2 - V$ must contain line segments unless V is constant. Its proof relies on delicate analysis based on detailed structure of Aubry–Mather sets in two dimensions and a rigidity result in Riemannian geometry (the Hopf conjecture). Besides, due to the highly nonlinear nature of the problem, efficient numerical schemes to compute $\overline{H}$ have yet to be found. We refer to [1–3,5–11,15] and the references therein for recent progress.

In this paper, we aim to investigate the relation between V and $\overline{H}$ from the perspective of the following inverse problem first formulated in [14].

Question 1. *Let $H \in C(\mathbb{R}^n)$ be a given coercive function, that is, $\lim_{|p|\to\infty} H(p) = +\infty$. Let $V_1, V_2 \in C(\mathbb{R}^n)$ be two given potential energy functions which are $\mathbb{Z}^n$-periodic. Let*