

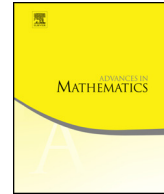


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# A rigidity result for effective Hamiltonians with 3-mode periodic potentials <sup>☆</sup>

Hung V. Tran <sup>a</sup>, Yifeng Yu <sup>b,\*</sup>

<sup>a</sup> Department of Mathematics, University of Wisconsin Madison, Van Vleck hall,  
480 Lincoln drive, Madison, WI 53706, USA

<sup>b</sup> Department of Mathematics, University of California, Irvine, 410G Rowland  
Hall, Irvine, CA 92697, USA

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## ABSTRACT

We continue studying an inverse problem in the theory of periodic homogenization of Hamilton–Jacobi equations proposed in [14]. Let  $V_1, V_2 \in C(\mathbb{R}^n)$  be two given potentials which are  $\mathbb{Z}^n$ -periodic, and  $\overline{H}_1, \overline{H}_2$  be the effective Hamiltonians associated with the Hamiltonians  $\frac{1}{2}|p|^2 + V_1, \frac{1}{2}|p|^2 + V_2$ , respectively. A main result in this paper is that, if the dimension  $n = 2$ , and each of  $V_1, V_2$  contains exactly 3 mutually non-parallel Fourier modes, then

$$\overline{H}_1 \equiv \overline{H}_2 \iff V_1(x) = V_2\left(\frac{x}{c} + x_0\right) \text{ for all } x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2,$$

for some  $c \in \mathbb{Q} \setminus \{0\}$  and  $x_0 \in \mathbb{T}^2$ . When  $n \geq 3$ , the scenario is slightly more subtle, and a complete description is provided for any dimension. These resolve partially a conjecture stated in [14]. Some other related results and open problems are also discussed.

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\* Corresponding author.

*E-mail addresses:* hung@math.wisc.edu (H.V. Tran), yyu1@math.uci.edu (Y. Yu).

### 1. Introduction

#### 1.1. Periodic homogenization and the inverse problem

We first describe the theory of periodic homogenization of Hamilton–Jacobi equations. For each length scale  $\varepsilon > 0$ , let  $u^\varepsilon \in C(\mathbb{R}^n \times [0, \infty))$  be the viscosity solution to

$$\begin{cases} u_t^\varepsilon + H(Du^\varepsilon) + V\left(\frac{x}{\varepsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u^\varepsilon(x, 0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here, the Hamiltonian  $H(p) - V(x)$  is of separable form with  $H \in C(\mathbb{R}^n)$ , which is coercive (i.e.,  $\lim_{|p| \rightarrow \infty} H(p) = +\infty$ ), and  $V \in C(\mathbb{R}^n)$ , which is  $\mathbb{Z}^n$ -periodic. The initial data  $g \in \text{BUC}(\mathbb{R}^n)$ , the set of bounded, uniformly continuous functions on  $\mathbb{R}^n$ .

It was shown in [13] that, in the limit as the length scale  $\varepsilon$  tends to zero,  $u^\varepsilon$  converges to  $u$  locally uniformly on  $\mathbb{R}^n \times [0, \infty)$ , and  $u$  solves the effective equation

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.2}$$

The effective Hamiltonian  $\overline{H} \in C(\mathbb{R}^n)$  is determined in a nonlinear way by  $H$  and  $V$  through the cell problems as follows. For each  $p \in \mathbb{R}^n$ , it was derived in [13] that there exists a unique constant  $c \in \mathbb{R}$  such that the following cell problem has a continuous viscosity solution

$$H(p + Dv) + V(x) = c \quad \text{in } \mathbb{T}^n, \tag{1.3}$$

where  $\mathbb{T}^n$  is the  $n$ -dimensional flat torus  $\mathbb{R}^n/\mathbb{Z}^n$ . We then denote by  $\overline{H}(p) := c$ .

During past decades, there have been tremendous progress and vast literature about the validity of homogenization and the well-posedness of cell problems in various generalized settings. Nevertheless, understanding theoretically how  $\overline{H}$  depends on the potential  $V$  remains a very challenging and still largely open problem even for the most basic case  $H(p) = \frac{1}{2}|p|^2$ . For a smooth periodic potential  $V$ , a deep result in [4] asserts that when  $n = 2$  and  $H(p) = \frac{1}{2}|p|^2$ , each non-minimum level curve of  $\overline{H}$  associated with  $\frac{1}{2}|p|^2 - V$  must contain line segments unless  $V$  is constant. Its proof relies on delicate analysis based on detailed structure of Aubry–Mather sets in two dimensions and a rigidity result in Riemannian geometry (the Hopf conjecture). Besides, due to the highly nonlinear nature of the problem, efficient numerical schemes to compute  $\overline{H}$  have yet to be found. We refer to [1–3, 5–11, 15] and the references therein for recent progress.

In this paper, we aim to investigate the relation between  $V$  and  $\overline{H}$  from the perspective of the following inverse problem first formulated in [14].

**Question 1.** *Let  $H \in C(\mathbb{R}^n)$  be a given coercive function, that is,  $\lim_{|p| \rightarrow \infty} H(p) = +\infty$ . Let  $V_1, V_2 \in C(\mathbb{R}^n)$  be two given potential energy functions which are  $\mathbb{Z}^n$ -periodic. Let*

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