Advances in Mathematics 334 (2018) 322–388

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Homogeneous families on trees and subsymmetric basic sequences ☆,☆☆

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A R T I C L E   I N F O

Article history:
Received 24 July 2016
Received in revised form 22 March 2018
Accepted 19 June 2018
Available online 30 June 2018
Communicated by Slawomir Solecki

MSC:
03E05
05D10
46B15
46B06
46B26

A B S T R A C T

We study density requirements on a given Banach space that guarantee the existence of subsymmetric basic sequences by extending Tsirelson's well-known space to larger index sets. We prove that for every cardinal κ smaller than the first Mahlo cardinal there is a reflexive Banach space of density κ without subsymmetric basic sequences. As for Tsirelson's space, our construction is based on the existence of a rich collection of homogeneous families on large index sets for which one can estimate the complexity on any given infinite set. This is used to describe detailedly the asymptotic structure of the spaces. The collections of families are of independent interest and their existence is proved inductively. The fundamental

☆ The first author was partially supported by FAPESP grants (2012/24463-7 and 2015/26654-2) and by CNPq grants (307942/2012-0 and 454112/2015-7). The second author was partially supported by FAPESP grant (2013/24827-1), by the Ministerio de Economía y Competitividad under grant MTM2012-31286 (Spain). The first and the third author were partially supported by USP-COFECUB grant (31466UC). The third author was also partially supported by grants from NSERC (455916) and CNRS (UMR7586).
☆☆ This research was partially done whilst the authors were visiting fellows at the Isaac Newton Institute for Mathematical Sciences in the programme "Mathematical, Foundational and Computational Aspects of the Higher Infinite" (HIF).

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https://doi.org/10.1016/j.aim.2018.06.008

Keywords:
Families of finite sets
Spreading models
Nonseparable Banach spaces

stepping up argument is the analysis of such collections of families on trees.

1. Introduction

Recall that a set of *indiscernibles* for a given structure $\mathcal{M}$ is a subset X with a total ordering $<$ such that for every positive integer n every two increasing n-tuples $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$ of elements of X have the same properties in $\mathcal{M}$. A simple way of finding an extended structure on κ without an infinite set of indiscernibles is as follows. Suppose that $\mathcal{F}$ is a family of finite subsets of κ that is compact, as a natural subset of the product space 2^κ, and *large*, that is, every infinite subset of κ has arbitrarily large subsets in $\mathcal{F}$. Let $\mathcal{M}_\mathcal{F}$ be the structure $(\kappa, (\mathcal{F} \cap [\kappa]^n)_n)$ that has κ as universe and that has infinitely many n-ary relations $\mathcal{F} \cap [\kappa]^n \subseteq [\kappa]^n$. It is easily seen that $\mathcal{M}_\mathcal{F}$ does not have infinite indiscernible sets.

While in set theory and model theory indiscernibility is a well-studied and unambiguous notion, in the context of the Banach space theory it has several versions, the most natural one being the notion of a *subsymmetric sequence* or a *subsymmetric set*. In a normed space $(X, \|\cdot\|)$, a sequence $(x_\alpha)_{\alpha \in I}$ indexed in an ordered set $(I, <)$ is called *C-subsymmetric* when

$$\|\sum_{j=1}^n a_j x_{\alpha_j}\| \leq C\|\sum_{j=1}^n a_j x_{\beta_j}\|$$

for every sequence of scalars $(a_j)_{j=1}^n$ and every $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ in I. When $C = 1$ this corresponds exactly to the notion of an indiscernible set and it is easily seen that this can always be assumed by renorming X with an appropriate equivalent norm. Another closely related notion is *unconditionality*. Recall that a sequence $(x_i)_{i \in I}$ in some Banach space is *C-unconditional* whenever

$$\|\sum_{i \in I} \theta_i a_i x_i\| \leq C\|\sum_{i \in I} a_i x_i\|$$

for every sequence of scalars $(a_i)_{i \in I}$ and every sequence $(\theta_i)_{i \in I}$ of signs. Of particular interest are the indiscernible coordinate systems, such as the Schauder basic sequences. The unit bases of the classical sequence spaces ℓ_p, $p \geq 1$ or c_0 (in any density) are subsymmetric and unconditional (in fact, symmetric, i.e. indiscernible by permutations) bases. Moreover, every basic sequence in one of these spaces has a symmetric subsequence. But this is not true in general: there are basic sequences without unconditional subsequences, the simplest example being the summing basis of c_0. However, it is more difficult to find a weakly-null basis without unconditional subsequences (B. Maurey and H.P. Rosenthal [17]). Now we know that there are Banach spaces without unconditional