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A variational principle for free semigroup actions

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ABSTRACT

In this paper we introduce a notion of measure theoretical entropy for a finitely generated free semigroup action and establish a variational principle when the semigroup is generated by continuous self maps on a compact metric space and has finite topological entropy. In the case of semigroups generated by Ruelle-expanding maps we prove the existence of equilibrium states and describe some of their properties. Of independent interest are the different ways we will present to compute the metric entropy and a characterization of the stationary measures.

1. Introduction

The concept of entropy, introduced into the realm of dynamical systems more than fifty years ago, has become an important ingredient in the characterization of the complexity of dynamical systems. The topological entropy reflects the complexity of the dynamical system and is an important invariant. The metric entropy of invariant measures turns

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out to be a surprisingly universal concept in ergodic theory since it appears in the study of different subjects, such as information theory, Poincaré recurrence or the analysis of either local or global dynamics. The classical variational principle for the topological entropy of continuous maps on compact metric spaces (see e.g. [26]) not only relates both concepts as allows one to describe the dynamics by means of a much richer combined review of both topological and ergodic aspects. An extension of the variational principle for more general group and semigroup actions face some nontrivial challenges. If, on the one hand, it runs into the difficulty to establish a suitable notion of topological complexity for the action, on the other hand, the existence of probability measures that are invariant by the (semi)group action is a rare event beyond the setting of finitely generated abelian groups. While the latter has become a major obstruction for the proof of variational principles for other group actions, leading contributions were established in the case of amenable and sofic groups [11,15,20].

We will consider random walks associated to finitely generated semigroup actions of continuous maps on a compact metric space X. More precisely, after fixing a finite set $\{g_1, g_2, \cdots, g_p\}$ of endomorphisms of X and taking the unilateral shift $\sigma\colon \Sigma_p^+ \to \Sigma_p^+$ defined on the space of sequences with values in $\{1, 2, \cdots, p\}$, endowed with a Borel σ-invariant probability measure $\mathbb{P}$, we associate to each $\omega = \omega_1\, \omega_2\, \cdots \in \Sigma_p^+$ the sequence of compositions $(g_{\omega_1}\, g_{\omega_2}\, \cdots g_{\omega_n})_{n \in \mathbb{N}}$. Our aim is to carry on the analysis, started in [9, 23], of the ergodic and statistical properties of these random compositions, and to set up a thermodynamic formalism. In this context common invariant measures seldom exist. Moreover, as we will discuss later on, stationary measures are not enough to describe the thermodynamic formalism of these semigroup actions.

One way to study a finitely generated free semigroup action is through the corresponding skew product $\mathcal{F}_G\colon \Sigma_p^+ \times X \to \Sigma_p^+ \times X$ defined by $\mathcal{F}_G(\omega, x) = (\sigma(\omega), g_{\omega_1}(x))$. This is an advantageous approach because $\mathcal{F}_G$ is a true dynamical system and, moreover, if each generator is a Ruelle-expanding map, then so is $\mathcal{F}_G$ and for it a complete formalism is known [24]. Yet, we are looking for intrinsic dynamical and ergodic concepts, the less dependent on the skew product the better. In [19], the authors proposed a notion of measure theoretical entropy for free semigroup actions and probability measures invariant by all the generators of the semigroup, but only a partial variational principle was obtained there (as happened in [5] and [10]). The major obstruction to use this strategy, in order to extend the ergodic theory known for a single expanding map, is the fact that probability measures which are invariant by all the generators of a free semigroup action may fail to exist. On the other hand, although stationary measures describe some of the statistics of the iteration of random transformations and are suited to the relativised variational principle proved for the skew product in [17] (using a notion of metric entropy different from ours and without any reference to semigroup actions), its role is hard to uncover without a major intervention of properties of the skew product $\mathcal{F}_G$. Moreover, as we will see, the set of stationary measures is scarce to describe in its full extent the ergodic attributes of a free semigroup action (though adequate in the setting of group