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Optimal existence classes and nonlinear-like dynamics in the linear heat equation in $\mathbb{R}^d$

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ABSTRACT

We analyse the behaviour of solutions of the linear heat equation in $\mathbb{R}^d$ for initial data in the classes $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ of Radon measures with $\int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2}\,\mathrm{d}|u_0| < \infty$. We show that these classes are optimal for local and global existence of non-negative solutions: in particular $\mathcal{M}_0(\mathbb{R}^d) := \cap_{\varepsilon>0}\mathcal{M}_\varepsilon(\mathbb{R}^d)$ consists of those initial data for which a solution of the heat equation can be given for all time using the heat kernel representation formula. We prove existence, uniqueness, and regularity results for such initial data, which can grow rapidly at infinity, and then show that they give rise to properties associated more often with nonlinear models. We demonstrate the finite-time blowup of solutions, showing that the set of blowup points is the complement of a convex set, and that given any closed convex set there is an initial condition whose solutions remain bounded precisely on this set at the 'blowup time'. We also show that wild oscillations are possible from non-negative initial data as $t \to \infty$ and that one can prescribe

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the behaviour of $u(0,t)$ to be any real-analytic function $\gamma(t)$ on $[0,\infty)$.

1. Introduction

In this paper we consider the linear heat equation posed on the whole space $\mathbb{R}^d$, with very general initial data, which may be either only locally integrable or even a Radon measure. For an appropriate class of initial data u_0, see e.g. [26], it is well known that solutions to this equation,

$$u_t - \Delta u = 0,\ x \in \mathbb{R}^d,\ t > 0, \qquad u(x,0) = u_0(x), \tag{1.1}$$

can be written using the heat kernel as

$$u(x,t) = S(t)u_0(x) := \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/4t} u_0(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d,\ t > 0. \tag{1.2}$$

It turns out that the behaviour of solutions in (1.2) is significantly affected by the way the mass of the initial data is distributed in space.

If the mass as $|x| \to \infty$ is not too large it is well known that the 'mass' of the initial data moves to infinity and the solutions decay to zero in suitable norms. For example, if $u_0 \in L^p(\mathbb{R}^d)$ for some $1 \le p < \infty$ then classical estimates ensure that

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \le (4\pi t)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p(\mathbb{R}^d)}, \quad \text{for every } t > 0 \text{ and } q \text{ with } p \le q \le \infty, \tag{1.3}$$

which in particular implies that all solutions converge uniformly to zero on the whole of $\mathbb{R}^d$. In particular, for $u_0 \in L^1(\mathbb{R}^d)$ since we also have

$$\int_{\mathbb{R}^d} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} u_0(y)\,\mathrm{d}y, \quad t > 0,$$

it follows that for such u_0 the total mass is preserved but (from (1.3)) the supremum tends to zero, i.e. the mass moves to infinity.

It is also known that as $t \to \infty$, solutions asymptotically resemble the heat kernel

$$K(x,t) = (4\pi t)^{-d/2} \mathrm{e}^{-|x|^2/4t},$$

see for example Section 1.1.4 in [13]. The faster the initial data decays as $|x| \to \infty$ the higher the order of the asymptotics of the solution that are described by the heat kernel, see e.g. [9].