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Linear and quadratic ranges in representation stability

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ABSTRACT

We prove two general results concerning spectral sequences of **FI**-modules. These results can be used to significantly improve stable ranges in a large portion of the stability theorems for **FI**-modules currently in the literature. We work this out in detail for the cohomology of configuration spaces where we prove a linear stable range and the homology of congruence subgroups of general linear groups where we prove a quadratic stable range. Previously, the best stable ranges known in these examples were exponential. Up to an additive constant, our

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work on congruence subgroups verifies a conjecture of Djament.

Contents

1. Introduction

FI-modules are a convenient framework for studying stability properties of sequences of symmetric group representations. An **FI**-module is a functor from the category of finite sets and injections to the category of $\mathbb{Z}$-modules. In this paper, we introduce two new techniques for proving stability results for graded sequences of **FI**-modules which yield improved stable ranges in many examples, including the cohomology of configuration spaces and the homology of congruence subgroups of general linear groups. In all applications, this grading will come from the standard grading on the (co)homology of a sequence of spaces or groups. By stability, we roughly mean a bound on the presentation degree in terms of the (co)homological degree. If there is such a bound which is linear in the (co)homological degree, we say that the sequence exhibits a linear stable range (similarly quadratic, exponential etc.).

While previous stability arguments focused on bounding the presentation degree, our proof strategy involves studying two other invariants of an **FI**-module. We call these invariants *stable degree* and *local degree* and show that these invariants are easier to control in spectral sequences than presentation degree. An **FI**-module over a ring $\mathbf{k}$ is a functor from **FI** to the category of abelian groups that factors though the category of $\mathbf{k}$-modules. Finitely generated **FI**-modules over fields have dimensions that are eventually equal to a polynomial. The stable degree of a finitely generated **FI**-module is equal to the degree of this polynomial and the local degree controls when these dimensions become equal to this polynomial.

Together, the stable degree and the local degree control the presentation degree of an **FI**-module (Proposition 3.1). Conversely, the presentation degree can be used to bound these invariants (Proposition 2.9 and Theorem 2.10). As a consequence, it is enough to bound the stable and the local degrees to bound the presentation degree and vice-versa. More precisely, we have the following quantitative result (Proposition 3.1):