Advances in Mathematics 333 (2018) 41–86

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

The Beilinson regulator is a map of ring spectra

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A R T I C L E   I N F O

Article history:
Received 17 January 2017
Received in revised form 26 April 2018
Accepted 1 May 2018
Available online 28 May 2018
Communicated by A. Blumberg

Keywords:
Beilinson regulator
K-theory
Absolute Hodge cohomology
Ring spectra
Motivic homotopy theory

A B S T R A C T

We prove that the Beilinson regulator, which is a map from K-theory to absolute Hodge cohomology of a smooth variety, admits a refinement to a map of E_∞-ring spectra in the sense of algebraic topology. To this end we exhibit absolute Hodge cohomology as the cohomology of a commutative differential graded algebra over $\mathbb{R}$. The associated spectrum to this CDGA is the target of the refinement of the regulator and the usual K-theory spectrum is the source. To prove this result we compute the space of maps from the motivic K-theory spectrum to the motivic spectrum that represents absolute Hodge cohomology using the motivic Snaith theorem. We identify those maps which admit an E_∞-refinement and prove a uniqueness result for these refinements.

Contents

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[1] The authors are supported by the SFB/CRC 1085 *Higher Invariants* (Universität Regensburg) funded by the DFG.

https://doi.org/10.1016/j.aim.2018.05.027
0001-8708/

1. Introduction

Let X be a smooth algebraic variety over $\mathbb{C}$. On the one hand, we can form its algebraic K-groups $K_*(X)$, which encode information about the symmetric monoidal category of vector bundles on X with respect to the direct sum. On the other hand, we have the Betti cohomology groups $H^*(X(\mathbb{C}), \mathbb{R})$, which carry a natural mixed $\mathbb{R}$-Hodge structure by [9]. In [1] Beilinson constructs a natural complex in the derived category of mixed $\mathbb{R}$-Hodge structures whose cohomology is the Betti cohomology with its mixed Hodge structure, and he defines absolute cohomology groups $H^*_{\text{abs. Hodge}}(X, \mathbb{R}(i))$ of X as Ext-groups of the Tate Hodge structure $\mathbb{R}(-i)$ and this complex. The absolute Hodge cohomology groups are the target of the Beilinson regulator, a natural homomorphism of graded groups

$$\mathrm{reg} : K_*(X) \to \bigoplus_{i \in \mathbb{N}} H^{2i-*}_{\text{abs. Hodge}}(X, \mathbb{R}(i)) \ . \tag{1}$$

The tensor product of vector bundles induces a commutative ring structure on $K_*(X)$, and the $\cup$-product in the Betti cohomology of X provides a commutative ring structure on $\bigoplus_{i \in \mathbb{N}} H^*_{\text{abs. Hodge}}(X, \mathbb{R}(i))$. It is known that the regulator is a homomorphism of rings [15, 2.35].

Our main motivation for the present paper is the application of the regulator to a multiplicative version of differential algebraic K-theory as discussed in the series of papers [2,5,6]. For this, one needs a more refined version of the regulator map: The algebraic K-groups $K_*(X)$ are defined as homotopy groups of an algebraic K-theory spectrum $\mathcal{K}(X)$ and the multiplication on the K-groups is induced by an E_∞-ring structure on this spectrum. Absolute Hodge cohomology on the other hand is defined as Ext-groups in the derived category of mixed Hodge complexes. We realize the absolute Hodge cohomology groups as the cohomology groups of a specific chain complex $\mathbf{IDR}(X)$ (Definition 3.3) consisting of differential forms. The usual wedge product of forms gives a multiplication on the chain level, i.e. it makes $\mathbf{IDR}(X)$ into a commutative differential graded algebra. Under the Eilenberg–MacLane equivalence H this commutative differential graded algebra induces an E_∞-ring spectrum $H(\mathbf{IDR}(X))$ whose homotopy groups are the cohomology groups of $\mathbf{IDR}(X)$ and therefore the absolute Hodge cohomology groups of X. For the application we have in mind, it is an important question whether the regulator