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On the planar dual Minkowski problem

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In this paper, we resolve the planar dual Minkowski problem, proposed by Huang et al. (2016) [31] for all positive indices without any symmetry assumption. More precisely, given any $q > 0$, and function f on $\mathbb{S}^1$, bounded by two positive constants, we show that there exists a convex body Ω in the plane, containing the origin in its interior, whose dual curvature measure $\widetilde{C}_q(\Omega, \cdot)$ has density f. In particular, if f is smooth, then $\partial\Omega$ is also smooth.

1. Introduction

The Minkowski type problems in convex geometry are characterisation problems of the differentials of geometric functions of convex bodies. The fundamental geometric functionals in the Brunn–Minkowski theory are the quermassintegrals, which include volume and surface area as special cases, while the area measures and the curvature

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measures can be viewed as the differentials of these quermassintegrals [47]. The associated Minkowski type problems then arise: that is to prescribe area measures and curvature measures. There is a large number of papers devoted to the study of these problems, see e.g. [2,16,25,27,42,46] and the references therein.

The dual Brunn–Minkowski theory emerged in the mid-1970s when Lutwak introduced the dual mixed volume [40]. The main geometric functionals in this theory are the dual quermassintegrals. Arising from the dual quermassintegrals is a new family of geometric measures, dual curvature measures $\widetilde{C}_q$ for $q \in \mathbb{R}$, discovered by Huang–Lutwak–Yang–Zhang in their recent seminal work [31]. Denote by $\mathcal{K}_0^n$ the set of all convex bodies (i.e., compact convex sets that have non-empty interior) in $\mathbb{R}^{n+1}$ containing the origin in their interiors. Associated to each convex body $\Omega \in \mathcal{K}_0^n$ are the support function $u = u_\Omega : \mathbb{S}^n \to \mathbb{R}$ and the radial function $r = r_\Omega : \mathbb{S}^n \to \mathbb{R}$, which are respectively defined by

$$u(x) = \max\{x \cdot z : \; z \in \Omega\},$$

and

$$r(\xi) = \max\{\lambda : \; \lambda\xi \in \Omega\}$$

Let $\vec{r}(\xi) = \vec{r}_\Omega(\xi) := r_\Omega(\xi)\xi$. Then obviously $\partial\Omega = \{\vec{r}(\xi) : \; \xi \in \mathbb{S}^n\}$. For $x \in \mathbb{S}^n$, the hyperplane

$$\ell_\Omega(x) = \{z \in \mathbb{R}^{n+1} : \; z \cdot x = u_\Omega(x)\}$$

is called the supporting hyperplane of Ω with the unit normal x. The spherical image $\nu = \nu_\Omega : \partial\Omega \to \mathbb{S}^n$ is then given by

$$\nu(z) = \{x \in \mathbb{S}^n : \; z \in \ell_\Omega(x)\}.$$

With the help of these notions, we can introduce two set-valued mappings, namely the radial Gauss mapping $\mathscr{A} = \mathscr{A}_\Omega$ and the reverse radial Gauss mapping $\mathscr{A}^* = \mathscr{A}^*_\Omega$ as follows: for any $\omega \subseteq \mathbb{S}^n$,

$$\begin{aligned} \mathscr{A}(\omega) &= \{\nu(\vec{r}(\xi)) : \; \xi \in \omega\}, \\ \mathscr{A}^*(\omega) &= \{\xi \in \mathbb{S}^n \; : \; \nu(\vec{r}(\xi)) \in \omega\}. \end{aligned} \tag{1.1}$$

The q-th dual curvature measure in [31] is defined as[1]

$$\widetilde{C}_q(\Omega, \omega) = \int\limits_{\mathscr{A}^*(\omega)} r^q(\xi) d\xi, \;\text{for}\; \omega \subseteq \mathbb{S}^n, \tag{1.2}$$

where $d\xi$ denotes the standard measure of $\mathbb{S}^n$.

[1] This definition differs a dimension constant with that in [31].