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Minkowski valuations under volume constraints [☆]

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A B S T R A C T

We provide a description of the space of continuous and translation invariant Minkowski valuations $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ for which there is an upper and a lower bound for the volume of $\Phi(K)$ in terms of the volume of the convex body K itself. Although no invariance with respect to a group acting on the space of convex bodies is imposed, we prove that only two types of operators appear: a family of operators having only cylinders over $(n-1)$-dimensional convex bodies as images, and a second family consisting essentially of 1-homogeneous operators. Using this description, we give improvements of some known characterization results for the difference body.

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1. Introduction

An inequality between two geometric quantities associated to a convex body is called *affine isoperimetric inequality* if the ratio of these two quantities is invariant under the action of all affine transformations of the convex body. Affine isoperimetric inequalities have always constituted an important part of convex geometry and have found numerous applications to different areas, such as functional analysis, partial differential equations, or geometry of numbers (see [41]). Moreover, affine isoperimetric inequalities are usually stronger than their Euclidean counterparts.

Three of the best known affine isoperimetric inequalities associated to operators between convex bodies are: the Rogers–Shephard inequality, associated to the difference body; the Busemann–Petty centroid inequality, associated to the centroid body; and the Petty projection and Zhang inequalities, associated to the projection body. One of the first and most relevant applications of these inequalities was given by Zhang [67], who obtained an affine version of the Sobolev inequality from (an extension of) the Petty projection inequality. Ten years later, Haberl and Schuster [26,27] generalized it to an asymmetric affine L_p-Sobolev inequality by using the characterization of the L_p-projection bodies previously obtained by Ludwig [35] in the context of the so-called L_p-Minkowski valuations. For further results in this direction we refer to [57, Section 10.15], [16,28,29,39,40,42–44,63], and references therein.

In the present paper, we initiate a study aiming at a deeper understanding of the relationship between affine isoperimetric inequalities and characterization results for Minkowski valuations, by taking the converse direction of Haberl and Schuster [26] and classifying, given an affine isoperimetric inequality, all continuous (and translation invariant) Minkowski valuations by which it is satisfied. In this paper, we focus on the affine isoperimetric inequality associated to the difference body operator.

We denote by $\mathcal{K}^n$ the space of convex and compact sets (convex bodies) in $\mathbb{R}^n$. The *difference body operator* $D : \mathcal{K}^n \longrightarrow \mathcal{K}^n$ is defined by

$$DK := K + (-K), \tag{1}$$

where $-K := \{x \in \mathbb{R}^n : -x \in K\}$ and $+$ denotes the Minkowski or vectorial sum. Notice that the ratio

$$\frac{V_n(DK)}{V_n(K)}$$

is invariant under affine transformations of $\mathbb{R}^n$ (here V_n denotes the n-dimensional volume). The affine isoperimetric inequalities associated to the difference body read as follows

$$2^n V_n(K) \leq V_n(DK) \leq \binom{2n}{n} V_n(K), \quad \forall K \in \mathcal{K}^n. \tag{RS}$$