Advances in Mathematics 333 (2018) 539–569

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Algebraic and o-minimal flows on complex and real tori ☆

Ya'acov Peterzil [a], Sergei Starchenko [b,*]

[a] *University of Haifa, Israel*
[b] *University of Notre Dame, United States of America*

A R T I C L E I N F O

Article history:
Received 14 September 2017
Received in revised form 27 March 2018
Accepted 25 May 2018
Available online 5 June 2018
Communicated by Slawomir Solecki

Keywords:
Algebraic flows
O-minimal flows
Orbit closure
O-minimal
Algebraically closed valued fields

A B S T R A C T

We consider the covering map $\pi : \mathbb{C}^n \to \mathbb{T}$ of a compact complex torus. Given an algebraic variety $X \subseteq \mathbb{C}^n$ we describe the topological closure of $\pi(X)$ in $\mathbb{T}$. We obtain a similar description when $\mathbb{T}$ is a real torus and $X \subseteq \mathbb{R}^n$ is a set definable in an o-minimal structure over the reals.

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☆ Both authors thank the Israel–US Binational Science Foundation for its support. The second author was supported by the NSF research grant DMS-1500671.

* Corresponding author.
E-mail addresses: kobi@math.haifa.ac.il (Y. Peterzil), sstarche@nd.edu (S. Starchenko).

https://doi.org/10.1016/j.aim.2018.05.040

1. Introduction

Let A be a complex abelian variety of dimension n, and let $\pi : \mathbb{C}^n \to A$ be its covering map. It follows from a theorem of Ax (see [1, Theorem 3]), that if $X \subseteq \mathbb{C}^n$ is an algebraic variety then the Zariski closure of $\pi(X)$ is a union of finitely many cosets of abelian subvarieties of A.

In [6,7], Ullmo and Yafaev attempt to characterize the *topological* closure of $\pi(X)$ in the above setting and also in the case that X is a set definable in an o-minimal expansion of the real field.

They prove a similar result to Ax's for algebraic curves (see [6, Theorem 2.4]: if $X \subseteq \mathbb{C}^n$ is an irreducible algebraic curve then the topological closure of $\pi(X)$ in A is

$$\text{cl}(\pi(X)) = \pi(X) \cup \bigcup_{k=1}^{m} Z_k,$$

where each Z_k is a real weakly special subvariety of A, namely a coset of a real Lie subgroup of A. They conjecture that the same is true for algebraic subvarieties $X \subseteq \mathbb{C}^n$ of arbitrary dimension.

In this article we give a full description of $\text{cl}(\pi(X))$ when X is an algebraic subvariety of $\mathbb{C}^n$ of arbitrary dimension and also when $X \subseteq \mathbb{R}^n$ is definable in an o-minimal structure over the reals and $\pi : \mathbb{R}^n \to \mathbb{T}$ is the covering map of a compact real torus.

As we show, the conjecture from [6] fails as stated (see Section 8) and we prove a modified version by showing that the frontier of $\pi(X)$ consists of finitely many families of real weakly special subvarieties. Our theorem holds for arbitrary compact complex tori and not only for abelian varieties.

Theorem 1.1. *Let $\pi : \mathbb{C}^n \to \mathbb{T}$ be the covering map of a compact complex torus and let X be an algebraic subvariety of $\mathbb{C}^n$. Then there are finitely many algebraic subvarieties*