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On Schur multiple zeta functions: A combinatoric generalization of multiple zeta functions

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A B S T R A C T

We introduce Schur multiple zeta functions which interpolate both the multiple zeta and multiple zeta-star functions of the Euler–Zagier type combinatorially. We first study their basic properties including a region of absolute convergence and the case where all variables are the same. Then, under an assumption on variables, some determinant formulas coming from theory of Schur functions such as the Jacobi–Trudi, Giambelli and dual Cauchy formula are established with the help of Macdonald's ninth variation of Schur functions. Moreover, we investigate the quasi-symmetric functions corresponding to the Schur multiple zeta functions. We obtain the similar results as above for them and, furthermore, describe the images of them by the antipode of the Hopf algebra of quasi-symmetric functions explicitly. Finally, we establish iterated integral representations of the

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Schur multiple zeta values of ribbon type, which yield a duality for them in some cases.

Contents

1. Introduction

The multiple zeta function and the multiple zeta-star function (MZF and MZSF for short) of the Euler–Zagier type are respectively defined by the series

$$\zeta(\boldsymbol{s}) = \sum_{m_1<\cdots<m_n} \frac{1}{m_1^{s_1}\cdots m_n^{s_n}}, \quad \zeta^\star(\boldsymbol{s}) = \sum_{m_1\le\cdots\le m_n} \frac{1}{m_1^{s_1}\cdots m_n^{s_n}},$$

where $\boldsymbol{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$. These series converge absolutely for $\Re(s_1), \dots, \Re(s_{n-1}) \ge 1$ and $\Re(s_n) > 1$ (see, e.g., [18] for more precise description about the region of absolute convergence). One easily sees that a MZSF can be expressed as a linear combination of MZFs, and vice versa. For instance,

$$\zeta^\star(s_1, s_2) = \zeta(s_1, s_2) + \zeta(s_1 + s_2),$$