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# On rigid compact complex surfaces and manifolds ☆,☆☆

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A B S T R A C T

This article investigates the subject of rigid compact complex manifolds. First of all we investigate the different notions of rigidity (local rigidity, global rigidity, infinitesimal rigidity, etale rigidity and strong rigidity) and the relations among them. Only for curves these notions coincide and the only rigid curve is the projective line. For surfaces we prove that a rigid surface which is not minimal of general type is either a Del Pezzo surface of degree $\geq 5$ or an Inoue surface. We give examples of rigid manifolds of dimension $n \geq 3$ and Kodaira dimensions 0, and $2 \leq k \leq n$. Our main theorem is that the Hirzebruch Kummer coverings of exponent $n \geq 4$ branched on a complete quadrangle are infinitesimally rigid. Moreover, we pose a number of questions.



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Deformation theory
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## 1. Introduction

The present investigation originated from some natural questions concerning the series of families of surfaces exhibited in [16] to provide counterexamples to a question posed by Fujita in 1982 (see also [17], [15]). The families depend on some integer invariants, the main one being an arbitrary integer $n$ coprime with 6, and for $n = 5$ they were first constructed in [3]: we shall refer to them as *BCD-surfaces.*

In case $n = 5$, the surfaces $S$ are ball quotients, hence they possess a Kähler metric with strongly negative curvature tensor, their universal covering $\tilde{S}$ is diffeomorphic to the Euclidean space $\mathbb{R}^4$, $\tilde{S}$ is a Stein manifold, and the surfaces $S$ are rigid in all possible senses (see the definitions given in section one).

It is natural to ask whether similar properties hold for the other BCD surfaces, in particular to ask about their rigidity. In fact, we prove the following

**Theorem 1.1.** *The BCD surfaces are infinitesimally rigid and rigid. As a consequence, since their Albanese map is a semistable fibration $\alpha : S \to B$ onto a curve $B$ of genus $b := \frac{1}{2}(n-1)$, and with fibres of genus $g = (n-1)$, we get a rigid curve $B$ inside the moduli stack $\overline{\mathfrak{M}_{n-1}}$ of stable curves of genus $(n-1)$.*

An interesting feature of the fibration is that all fibres are smooth, except three fibres which are the union of two smooth curves of genus $b$ intersecting transversally in one point, so that the Jacobians of all the fibres are principally polarized Abelian varieties, and $B$ yields a complete curve inside $\mathfrak{A}_{n-1}$; it is an interesting question whether this curve is also rigid inside $\mathfrak{A}_{n-1}$ (see [40], [19] and [24] for related questions).

Suspicion of rigidity came from the observation that all deformations of BCD also have an Albanese map which is a fibration onto a curve of genus $b$ with exactly three singular fibres, of the same type as described above. The proof of rigidity however follows another path: first of all we observe that BCD surfaces admit as a finite unramified covering the