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Information-theoretic thresholds from the cavity method

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ABSTRACT

Vindicating a sophisticated but non-rigorous physics approach called the cavity method, we establish a formula for the mutual information in statistical inference problems induced by random graphs and we show that the mutual information holds the key to understanding certain important phase transitions in random graph models. We work out several concrete applications of these general results. For instance, we pinpoint the exact condensation phase transition in the Potts antiferromagnet on the random graph, thereby improving prior approximate results (Contucci et al., 2013) [34]. Further, we prove the conjecture from Krzakala et al. (2007) [55]

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about the condensation phase transition in the random graph coloring problem for any number $q \geq 3$ of colors. Moreover, we prove the conjecture on the information-theoretic threshold in the disassortative stochastic block model (Decelle et al., 2011) [35]. Additionally, our general result implies the conjectured formula for the mutual information in Low-Density Generator Matrix codes (Montanari, 2005) [73].

1. Introduction

Since the late 1990's physicists have studied models of spin systems in which the geometry of interactions is determined by a sparse random graph in order to better understand "disordered" physical systems such as glasses or spin glasses [67,68,72]. To the extent that the sparse random graph induces an actual geometry on the sites, such "diluted mean-field models" provide better approximations to physical reality than models on the complete graph such as the Curie–Weiss or the Sherrington–Kirkpatrick model [66]. But in addition, and perhaps more importantly, as random graph models occur in many branches of science, the physics ideas have since led to intriguing predictions on an astounding variety of important problems in mathematics, computer science, information theory, and statistics. Prominent examples include the phase transitions in the random k-SAT and random graph coloring problems [70,88], both very prominent problems in combinatorics, error correcting codes [66], compressed sensing [89], and the stochastic block model [35], a classical statistical inference problem.

The thrust of this work goes as follows. In many problems random graphs are either endemic or can be introduced via probabilistic constructions. As an example of the former think of the stochastic block model, where the aim is to recover a latent partition from a random graph. For an example of the latter, think of low density generator matrix 'LDGM' codes, where by design the generator matrix is the adjacency matrix of a random bipartite graph. To models of either type physicists bring to bear the *cavity method* [69], a comprehensive tool for studying random graph models, to put forward predictions on phase transitions and the values of key quantities. The cavity method comes in two installments: the replica symmetric version, whose mainstay is the Belief Propagation messages passing algorithm, and the more intricate replica symmetry breaking version, but it has emerged that the replica symmetric version suffices to deal with many important models.

Yet the cavity method suffers an unfortunate drawback: it is utterly non-rigorous. In effect, a substantial research effort in mathematics has been devoted to proving specific conjectures based on the physics calculations. Success stories include the ferromagnetic Ising model and Potts models on the random graph [38,37], the exact k-SAT threshold for large k [31,42], the condensation phase transition in random graph coloring [20], work on the stochastic block model [62,77,78] and terrific results on error correcting codes [47].