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Introducing symplectic billiards

Peter Albers [a,*], Serge Tabachnikov [b]

[a] *Mathematisches Institut, Ruprecht-Karls-Universität Heidelberg, Germany*
[b] *Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA*

A R T I C L E   I N F O

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A B S T R A C T

In this article we introduce a simple dynamical system called symplectic billiards. As opposed to usual/Birkhoff billiards, where length is the generating function, for symplectic billiards symplectic area is the generating function. We explore basic properties and exhibit several similarities, but also differences of symplectic billiards to Birkhoff billiards.

1. Introduction

Birkhoff billiard describes the motion of a free particle in a domain: when the particle hits the boundary, it reflects elastically so that the tangential component of its velocity remains the same and the normal component changes the sign. In the plane, this is the

* Corresponding author.
E-mail addresses: palbers@mathi.uni-heidelberg.de (P. Albers), tabachni@math.psu.edu (S. Tabachnikov).

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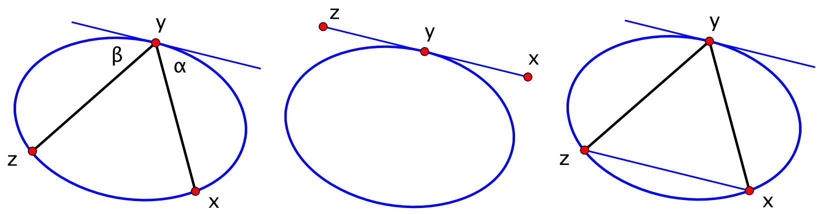

Fig. 1. Left, the usual billiard reflection: xy reflects to yz if $\alpha = \beta$. Middle, the outer billiard reflection: x reflects to z if $|xy| = |yz|$. Right, the symplectic billiard reflection: xy reflects to yz if xz is parallel to the tangent line of the curve at point y.

familiar law of geometric optics: the angle of incidence equals the angle of reflection, see Fig. 1.

This reflection law has a variational formulation: if the points x and z are fixed, the position of the point y on the billiard curve is determined by the condition that the length $|xy| + |yz|$ is extremal. In particular, periodic billiard trajectories are inscribed polygons of extremal perimeter length.

Another well-studied dynamical system is outer billiard, one such transformation in the exterior of a convex curve also depicted in Fig. 1. Outer billiard admits a variational formulation as well: periodic outer billiard orbits are circumscribed polygons of extremal area.

It is natural to consider two other planar billiards: inner with area, and outer with the length as the generating functions. The variational formulation of the reflection depicted in Fig. 1 on the right is as follows: if the points x and z are fixed, the position of the point y on the billiard curve is determined by the condition that the (unsigned) area of the triangle xyz is maximal. It would be natural to call this *area billiard*[1]; due to higher-dimensional considerations, we prefer the term *symplectic billiard*.

Analytically, the generating function of the symplectic billiard map is $\omega(x, y)$, where ω is the area form, as opposed to $|xy|$, the generating function of the usual billiard map.

The aim of this paper is to introduce symplectic billiards and to make the first steps in their study. We believe that a system that has such a simple and natural definition should have interesting properties; in particular, one wants to learn which familiar properties of the usual inner (Birkhoff) and outer billiards are specific to them, and which extend to the symplectic billiards and, perhaps, other similar systems.

One can also define symplectic billiard in a convex domain with smooth boundary in a linear symplectic space $(\mathbb{R}^{2n}, \omega)$. Let M be its boundary. To define a reflection similarly to the planar case, one needs to choose a tangent direction at every point of M. This is canonically provided by the symplectic structure: the tangent line at $x \in M$ is the characteristic direction $R(x) = \ker \omega|_{T_x M}$, that is, the kernel of the restriction of the symplectic form ω on the tangent hyperplane $T_x M$. (A similar idea is used in the

[1] A competing name would be *affine billiard* since this system commutes with affine transformations of the plane.