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Congruence lattices of finite diagram monoids

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ABSTRACT

We give a complete description of the congruence lattices of the following finite diagram monoids: the partition monoid, the planar partition monoid, the Brauer monoid, the Jones monoid (also known as the Temperley–Lieb monoid), the Motzkin monoid, and the partial Brauer monoid. All the congruences under discussion arise as special instances of a new construction, involving an ideal I, a retraction $I \to M$ onto the minimal ideal, a congruence on M, and a normal subgroup of a maximal subgroup outside I.

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1. Introduction

A congruence on a semigroup S is an equivalence relation that is compatible with the multiplicative structure of S. The role played by congruences in semigroup theory (and general algebra) is analogous to that of normal subgroups in group theory, and ideals in ring theory: they are precisely the kernels of homomorphisms, and they govern the formation of quotients. The set $\mathrm{Cong}(S)$ of all congruences of a semigroup S forms an (algebraic) lattice under inclusion, known as the congruence lattice of S.

The study of congruences has always been one of the corner-stones of semigroup theory, and a major strand in this direction has been the description of the congruence lattices of specific semigroups or families of semigroups. In his influential 1952 article [26], Mal'cev described the congruences on the full transformation monoid $\mathcal{T}_n$. Analogues for other classical monoids followed: the monoid of $n \times n$ matrices over a field (Mal'cev [27]), the symmetric inverse monoid $\mathcal{I}_n$ (Liber [25]), the partial transformation semigroup $\mathcal{PT}_n$ (Sutov [37]), and many others subsequently; see for example [2,16,29,38]. A contemporary account of these results for $\mathcal{T}_n$, $\mathcal{PT}_n$, and $\mathcal{I}_n$ can be found in [18, Section 6.3]. It turns out that in each of these cases, the congruence lattice is a chain whose length is a linear function of n. An even more recent work is the paper [3] by Araújo, Bentz and Gomes, describing the congruences in various direct products of transformation monoids.

In the current article, we undertake a study of congruence lattices of diagram monoids. These monoids arise naturally in the study of diagram algebras, a class of algebras with origins in theoretical physics and representation theory. Key examples are the Temperley–Lieb algebras [5,30,39], Brauer algebras [6,32], partition algebras [19,22,30] and Motzkin algebras [4]. These diagram algebras are defined by means of diagrammatic basis elements, and are all twisted semigroup algebras [40] of a corresponding diagram monoid, such as the Jones, Brauer, partition or Motzkin monoid; see Section 2 for the definitions of these monoids and others.

There are many important connections between diagram and transformation monoids. For one thing, the partition monoid $\mathcal{P}_n$ contains copies of the full transformation monoid $\mathcal{T}_n$ and the symmetric inverse monoid $\mathcal{I}_n$. In addition, many of the semigroup-theoretic properties of transformation monoids also hold for diagram monoids. For example, in each of the above-mentioned diagram monoids, the ideals form a chain with respect to containment. Furthermore, the idempotent-generated subsemigroup coincides with the singular part of the Brauer and partition monoids [12,28], and the proper ideals of the Jones, Brauer and partition monoids are idempotent-generated [14]. When it comes to congruences, however, it turns out that the parallels are simultaneously less tight and more subtle.

This paper arose as a consequence of some initial computational experiments with the Semigroups package for the computer algebra system GAP [34]. Using newly developed algorithms for working with congruences, we were able to compute the congruence lattices of several diagram monoids, including the partition, Brauer and Jones monoids of relatively low degree. The results of these computations were surprising at the time.