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An approximate version of a conjecture of Aharoni and Berger ☆

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ABSTRACT

Aharoni and Berger conjectured that in every proper edge-colouring of a bipartite multigraph by n colours with at least $n+1$ edges of each colour there is a rainbow matching using every colour. This conjecture generalizes a longstanding problem of Brualdi and Stein about transversals in Latin squares. Here an approximate version of the Aharoni–Berger Conjecture is proved—it is shown that if there are at least $n+o(n)$ edges of each colour in a proper n-edge-colouring of a bipartite multigraph then there is a rainbow matching using every colour.

1. Introduction

The research in this paper is motivated by some old problems about transversals in Latin squares. Recall that a Latin square of order n is an $n \times n$ array filled with n

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different symbols, where no symbol appears in the same row or column more than once. A transversal in a Latin square of order n is a set of n entries such that no two entries are in the same row, same column, or have the same symbol. It is easy to see that not every Latin square has a transversal (for example the unique 2×2 Latin square has no transversal.) However, it is possible that every Latin square contains a large *partial transversal*. Here, a partial transversal of size m means a set of m entries such that no two entries are in the same row, same column, or have the same symbol. The study of transversals in Latin squares goes back to Euler who studied orthogonal Latin squares i.e. order n Latin squares which can be decomposed into n disjoint transversals. For a survey of transversals in Latin squares, see [16].

There are several closely related, old, and difficult conjectures which say that Latin squares should have large partial transversals. The first of these is a conjecture of Ryser that every Latin square of odd order contains a transversal [14]. Brualdi conjectured that every Latin square contains a partial transversal of size $n-1$ (see [6].) Stein independently made the stronger conjecture that every $n \times n$ array filled with n symbols, each appearing exactly n times contains a partial transversal of size $n-1$ [15]. Because of the similarity of the above two conjectures, the following is often referred to as "the Brualdi–Stein Conjecture".

Conjecture 1.1 *(Brualdi and Stein, [6,15]). Every $n \times n$ Latin square has a partial transversal of size $n-1$.*

In this paper we will study a generalization of the Brualdi–Stein Conjecture to the setting of rainbow matchings in properly coloured bipartite multigraphs. How are these related? There is a one-to-one correspondence between $n \times n$ Latin squares and proper edge colourings of $K_{n,n}$ with n colours. Indeed consider a Latin square S whose set of symbols is $\{1, \dots, n\}$ with the i, j symbol $S_{i,j}$. To S we associate an edge-colouring of $K_{n,n}$ with the colours $\{1, \dots, n\}$, by setting $V(K_{n,n}) = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ and letting the edge between x_i and y_j receive colour $S_{i,j}$. Notice that this colouring is proper i.e. adjacent edges receive different colours. Recall that a *matching* in a graph is a set of disjoint edges. We call a matching *rainbow* if all of its edges have different colours. It is easy to see that partial transversals in the Latin square S correspond to rainbow matchings in the corresponding coloured $K_{n,n}$. Thus the Brualdi–Stein Conjecture is equivalent to the statement that "in any proper n-edge-colouring of $K_{n,n}$, there is a rainbow matching of size $n-1$." Once the conjecture is phrased in this form, one begins to wonder whether large rainbow matchings should exist in more general coloured graphs. Aharoni and Berger made the following generalization of the Brualdi–Stein Conjecture.

Conjecture 1.2 *(Aharoni and Berger, [1]). Let G be a properly edge-coloured bipartite multigraph with n colours having at least $n+1$ edges of each colour. Then G has a rainbow matching using every colour.*