Advances in Mathematics 332 (2018) 1–33

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Dihedral branched covers of four-manifolds

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ARTICLE INFO

Article history:
Received 9 January 2017
Received in revised form 29 March 2018
Accepted 1 April 2018
Available online xxxx
Communicated by Tony Pantev

MSC:
primary 57M12
secondary 57M27

Keywords:
branched cover
signature
four-manifold
singularity
knot

ABSTRACT

Given a closed oriented PL four-manifold X and a closed surface B embedded in X with isolated cone singularities, we give a formula for the signature of an irregular dihedral cover of X branched along B. For X simply-connected, we deduce a necessary condition on the intersection form of a simply-connected irregular dihedral branched cover of (X, B). When the singularities on B are two-bridge slice, we prove that the necessary condition on the intersection form of the cover is sharp. For X a simply-connected PL four-manifold with non-zero second Betti number, we construct infinite families of simply-connected PL manifolds which are irregular dihedral branched coverings of X. Given two four-manifolds X and Y whose intersection forms are odd, we obtain a necessary and sufficient condition for Y to be homeomorphic to an irregular dihedral p-fold cover of X, branched over a surface with a two-bridge slice singularity.

1. Introduction

The classification of all branched covers over a given base is a subject dating back to Alexander, who proved that every closed orientable PL n-manifold is a PL branched

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https://doi.org/10.1016/j.aim.2018.04.016

cover of S^n [1]. Alexander's branching sets are PL subcomplexes of the sphere; he concludes little else about them. Since 1920, the natural question of how complicated the branching set needs to be, and how many sheets are needed, in order to realize all manifolds in a given dimension as branched covers of the sphere, has received much interest – see, for instance, [3] and references therein. It is a famous theorem in dimension 3 that three-fold dihedral covers branched along knots suffice [18], [19], [26]. The question is considerably more subtle in dimension four. Piergallini and Iori, among others, have studied the minimal degree needed to realize all closed oriented PL four-manifolds as covers of the sphere. The branching sets they consider are either immersed PL submanifolds with transverse self-intersections or embedded and non-singular PL surfaces. Piergallini proved in [32] that every closed oriented PL four-manifold is a four-fold cover of S^4 branched over a over a transversally immersed PL surface. He and Iori later refined this result to show in [20] that singularities can be removed by stabilizing to a five-fold cover. In light of these universal realization theorems, one might wish for equally general methods for obtaining explicit descriptions of the branching sets needed to realize particular PL four-manifolds as a five-fold covers of S^4. It would also be of interest to better understand the trade-off between simplifying the branching set and increasing the degree of a cover. Most recently, Piergallini and Zuddas [33] showed that closed oriented *topological* four-manifolds are also five-fold covers of the sphere, if one allows for "wild" branching sets with potentially very pathological topology near isolated points. Still, the complexity of the branching sets near the wild points retains an air of mystery.

We assume a complementary approach, taking the point of view of studying all possible covers over a given base X *in terms of the branching set* and its embedding into the base. As seen from the main theorem of [40], if Y is a cover of S^4 branched over a closed oriented non-singular embedded surface, then the signature of Y must be zero. Thus, for example, the existing results on five-fold covers of the four-sphere implicitly make use of nonorientable branching sets. In contrast, the constructions presented here make use of branching sets that are oriented surfaces, embedded in the base piecewise linearly except for finitely many cone singularities. These ideas have led to new examples of branched covers of S^4 and applications to the Slice–Ribbon Conjecture [5]. We work with irregular dihedral covers (Definition 1.1), which constitute the most direct generalization of the three-dimensional results of Hilden, Hirsch and Montesinos as well as four-dimensional results of Montesinos [27]. Dihedral covers are also the "simplest" three-fold covers which give rise to interesting examples where the branching sets are singularly embedded (see Remark 2.2).

Our results are not restricted to branched covers of the sphere but apply to any closed oriented four-manifold base. Given an irregular dihedral branched cover $f : Y \to X$ between two simply-connected oriented four-manifolds X and Y, we relate the intersection forms of X and Y via f. Singularities for us play a central role, and we compute the signature of a branched cover in terms of data about the branching set and its singularity.

We begin by defining the type of covers and singularities considered. Throughout, D_p denotes the dihedral group of order $2p$ and p is odd.