Advances in Mathematics 332 (2018) 57–84

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

On the L_p dual Minkowski problem

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ARTICLE INFO

Article history:
Received 4 June 2017
Received in revised form 1 April 2018
Accepted 12 April 2018
Communicated by the Managing Editors of AIM

MSC:
52A40
52A38
35J96
35J20

Keywords:
L_p dual Minkowski problem
L_p Minkowski problem
Dual Minkowski problem
Degenerate Monge–Ampère equation
Regularity

ABSTRACT

The L_p dual curvature measure was introduced by Lutwak, Yang & Zhang in an attempt to unify the L_p Brunn–Minkowski theory and the dual Brunn–Minkowski theory. The characterization problem for L_p dual curvature measure, called the L_p dual Minkowski problem, is a fundamental problem in this unifying theory. The L_p dual Minkowski problem contains the L_p Minkowski problem and the dual Minkowski problem, two major problems in modern convex geometry that remain open in general. In this paper, existence results on the L_p dual Minkowski problem in the weak sense will be provided. Moreover, existence and uniqueness of the solution in the smooth category will also be demonstrated.

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[1] Research of the first author was supported by the National Science Fund for Distinguished Young Scholars (No. 11625103) and the Fundamental Research Funds for the Central Universities.

https://doi.org/10.1016/j.aim.2018.05.002
0001-8708/

1. Introduction

The classical Brunn–Minkowski theory focused on studying geometric invariants such as quermassintegrals (which include volume, surface area, and mean width) and geometric measures such as area measures and Federer's curvature measures. Among these measures are surface area measure and (Aleksandrov's) integral curvature, two most studied measures in the Brunn–Minkowski theory. The Minkowski problem and the Aleksandrov problem characterizing these two measures are influential problems not only in geometric analysis, but also in the theory of fully nonlinear partial differential equations.

The L_p Brunn–Minkowski theory and the dual Brunn–Minkowski theory are two theories that fundamentally extended the classical Brunn–Minkowski theory. The last two decades saw the rapid development of these two theories and they are now becoming the center focus of modern convex geometry.

The L_p Brunn–Minkowski theory came to life when Lutwak [49,50] introduced L_p surface area measure. When $p = 1$, the L_p surface area measure is the classical surface area measure. Since its introduction, the family of L_p surface area measure has quickly become the topic of many influential works. The L_p Minkowski problem is the problem of prescribing L_p surface area measure, which greatly generalizes the classical Minkowski problem. The family of L_p Minkowski problems contains important singular *unsolved* cases such as the logarithmic Minkowski problem (see Böröczky, Lutwak, Yang & Zhang [10]) and the centro-affine Minkowski problem.

The dual Brunn–Minkowski theory was introduced by Lutwak (see Schneider [58]) in the 1970s. The dual Brunn–Minkowski theory has been most effective in answering questions related to intersections. One major triumph of the dual Brunn–Minkowski theory is tackling the famous Busemann–Petty problem, see Gardner [21], Gardner, Koldobsky & Schlumprecht [23], Koldobsky [38–40], Lutwak [48], and Zhang [67]. Over the years, the dual theory has produced numerous profound concepts and results. See Gardner [22] and Schneider [58] for an overview of the theory. The dual theory makes extensive use of techniques from harmonic analysis. Recently, the dual Brunn–Minkowski theory took a huge step forward when Huang, Lutwak, Yang & Zhang [33] discovered the family of fundamental geometric measures—called dual curvature measures—in the dual theory. These measures are dual to Federer's curvature measures and are expected to play the same important role as area measures and curvature measures in the Brunn–Minkowski theory. The dual Minkowski problem is the problem of prescribing dual curvature measures. The dual Minkowski problem not only contains critical problems such as the logarithmic Minkowski problems and the Aleksandrov problem (prescribing curvature measure) as special cases, but also introduces intrinsic PDEs—something long missing—to the dual Brunn–Minkowski theory. The dual Minkowski problem, while still largely open, has been studied in [8,12,30,33,68,69].

A recent surprising discovery by Lutwak, Yang & Zhang [53] revealed that there exists a unifying theory that includes the classical Brunn–Minkowski theory, the L_p Brunn–Minkowski theory, and the dual Brunn–Minkowski theory. The latter two were never