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Shuffle-compatible permutation statistics ☆

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ABSTRACT

Since the early work of Richard Stanley, it has been observed that several permutation statistics have a remarkable property with respect to shuffles of permutations. We formalize this notion of a shuffle-compatible permutation statistic and introduce the shuffle algebra of a shuffle-compatible permutation statistic, which encodes the distribution of the statistic over shuffles of permutations. This paper develops a theory of shuffle-compatibility for descent statistics—statistics that depend only on the descent set and length—which has close connections to the theory of P-partitions, quasisymmetric functions, and noncommutative symmetric functions. We use our framework to prove that many descent statistics are shuffle-compatible and to give explicit descriptions of their shuffle algebras, thus unifying past results of Stanley, Gessel, Stembridge, Aguiar–Bergeron–Nyman, and Petersen.

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1. Introduction

We say that $\pi = \pi_1\pi_2\cdots\pi_n$ is a *permutation* of *length* n (or an n-*permutation*) if it is a sequence of n distinct letters—not necessarily from 1 to n—in $\mathbb{P}$, the set of positive integers. For example, $\pi = 47381$ is a permutation of length 5. Let $|\pi|$ denote the length of a permutation π and let $\mathfrak{P}_n$ denote the set of all permutations of length n.[2]

A *permutation statistic* (or *statistic*) st is a function defined on permutations such that $\operatorname{st}(\pi) = \operatorname{st}(\sigma)$ whenever π and σ are permutations with the same relative order.[3] Three classical examples of permutation statistics are the descent set Des, the descent number des, and the major index maj. We say that $i \in [n-1]$ is a *descent* of $\pi \in \mathfrak{P}_n$ if $\pi_i > \pi_{i+1}$. Then the *descent set*

$$\operatorname{Des}(\pi) := \{\, i \in [n-1] : \pi_i > \pi_{i+1} \,\}$$

[2] In Section 2, we will in a few instances consider permutations with a letter 0. We note that, in these cases, every property of permutations that is used still holds when 0 is allowed to be a letter.

[3] Define the *standardization* of an n-permutation π to be the permutation of $[n]$ obtained by replacing the ith smallest letter of π with i for i from 1 to n. Then two permutations are said to *have the same relative order* if they have the same standardization.