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Hyperspaces of smooth convex bodies up to congruence

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ABSTRACT

We determine the homeomorphism type of the hyperspace of positively curved C^∞ convex bodies in $\mathbb{R}^n$, and derive various properties of its quotient by the group of Euclidean isometries. We make a systematic study of hyperspaces of convex bodies that are at least C^1. We show how to destroy the symmetry of a family of convex bodies, and prove that this cannot be done modulo congruence.

1. Introduction

A *hyperspace of* $\mathbb{R}^n$ is a set of compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff metric, and two subsets are *congruent* if they lie in the same orbit of $\mathrm{Iso}(\mathbb{R}^n)$, the group of Euclidean isometries. To avoid trivialities *we assume that* $n \geq 2$.

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A *convex body in* $\mathbb{R}^n$ is a compact convex set with nonempty interior. A function is $C^{k,\alpha}$ if its kth partial derivatives are α-Hölder for $\alpha \in (0,1]$ and continuous for $\alpha = 0$ where as usual C^k means $C^{k,0}$, see [28] and [9, Section 2] for background. A $C^{k,\alpha}$ *convex body* is a convex body whose boundary is a $C^{k,\alpha}$ submanifold of $\mathbb{R}^n$. Any convex body is $C^{0,1}$ because convex functions are locally Lipschitz.

There is an established framework for studying topological properties of hyperspaces, and e.g., the homeomorphism types of the following hyperspaces of $\mathbb{R}^n$ are known: convex compacta [32], convex bodies [7], convex polyhedra [10, Exercise 4.3.7], strictly convex bodies [12], convex compacta of constant width [11,13,8].

One goal of this paper is to add to this list the hyperspace of C^∞ convex bodies of positive Gaussian curvature.

Another goal is to study certain hyperspaces that are not closed under Minkowski sum, e.g., the results of this paper are used in [14] to determine the homeomorphism type of a hyperspace of $\mathbb{R}^3$ whose $\mathrm{Iso}(\mathbb{R}^3)$-quotient is homeomorphic to the Gromov–Hausdorff space of C^∞ nonnegatively curved 2-spheres.

Let $\mathcal{K}_s$ be the hyperspace of convex compacta in $\mathbb{R}^n$ with Steiner point at the origin, and let $\mathcal{B}_p$ be the hyperspace of C^∞ convex bodies in $\mathcal{K}_s$ with boundary of positive Gaussian curvature. Placing the Steiner point at the origin is mainly a matter of convenience. In particular, the space of all convex compacta $\mathcal{K}$ in $\mathbb{R}^n$ is homeomorphic to $\mathcal{K}_s \times \mathbb{R}^n$, and the orbit spaces $\mathcal{K}/\mathrm{Iso}(\mathbb{R}^n)$, $\mathcal{K}_s/O(n)$ are homeomorphic, see (4.2) and Lemma 4.3.

Consider the *Hilbert cube* $Q = [-1,1]^\omega$ and its *radial interior*

$$\Sigma = \{(t_i)_{i\in\omega} \text{ in } Q \,:\, \sup_{i\in\omega} |t_i| < 1\}.$$

The superscript ω refers to the product of countably many copies of a space. We have a canonical inclusion $\Sigma^\omega \subset Q^\omega$. Clearly Q^ω and Q are homeomorphic. Also Σ is σ-compact while Σ^ω is not. Here is the main result of this paper.

Theorem 1.1. *Given a point $*$ in $Q^\omega \setminus \Sigma^\omega$ there exists a homeomorphism $\mathcal{K}_s \to Q^\omega \setminus \{*\}$ that takes $\mathcal{B}_p$ onto Σ^ω.*

The proof of Theorem 1.1 verifies the assumptions of recognition theorems for spaces modeled on Q and Σ^ω as described e.g., in [10]. This involves various convex geometry techniques, as well as a method developed in [9].

Yet another objective of this paper is to study the topology of $\mathcal{K}_s$ and $\mathcal{B}_p$ modulo congruence, see [36,4,2,7,1] for related results.

Set $\boldsymbol{\mathcal{K}}_s = \mathcal{K}_s/O(n)$ and $\boldsymbol{\mathcal{B}}_p = \mathcal{B}_p/O(n)$ with the quotient topology. Let $\mathring{\mathcal{K}}_s$, $\mathring{\mathcal{B}}_p$ be the subspaces of $\mathcal{K}_s$, $\mathcal{B}_p$, respectively, consisting of convex compacta with trivial stabilizer in $O(n)$, and let $\mathring{\boldsymbol{\mathcal{K}}}_s$, $\mathring{\boldsymbol{\mathcal{B}}}_p$ be their $O(n)$-orbit spaces. By the Slice Theorem $\mathring{\boldsymbol{\mathcal{K}}}_s$ is open in $\boldsymbol{\mathcal{K}}_s$ while the orbit maps $\mathring{\mathcal{K}}_s \to \mathring{\boldsymbol{\mathcal{K}}}_s$ and $\mathring{\mathcal{B}}_p \to \mathring{\boldsymbol{\mathcal{B}}}_p$ are principal $O(n)$-bundles. Some other properties of the spaces are summarized below.