Advances in Mathematics 332 (2018) 311–348

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Zeros of the deformed exponential function

Liuquan Wang [a], Cheng Zhang [b,*]

[a] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, Hubei, People's Republic of China

[b] Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, United States

ARTICLE INFO

Article history:
Received 7 January 2018
Received in revised form 4 April 2018
Accepted 20 April 2018
Available online xxxx
Communicated by George Andrews

MSC:
primary 30C15, 11B83, 41A60
secondary 11M36, 34K06

Keywords:
Deformed exponential function
Asymptotic expansion
Eisenstein series
Bernoulli numbers

ABSTRACT

Let $f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}q^{n(n-1)/2}x^n$ $(0<q<1)$ be the deformed exponential function. It is known that the zeros of $f(x)$ are real and form a negative decreasing sequence (x_k) $(k\geq 1)$. We investigate the complete asymptotic expansion for x_k and prove that for any $n\geq 1$, as $k\to\infty$,

$$x_k=-kq^{1-k}\Big(1+\sum_{i=1}^{n}C_i(q)k^{-1-i}+o(k^{-1-n})\Big),$$

where $C_i(q)$ are some q series which can be determined recursively. We show that each $C_i(q)\in\mathbb{Q}[A_0,A_1,A_2]$, where $A_i=\sum_{m=1}^{\infty}m^i\sigma(m)q^m$ and $\sigma(m)$ denotes the sum of positive divisors of m. When writing C_i as a polynomial in A_0, A_1 and A_2, we find explicit formulas for the coefficients of the linear terms by using Bernoulli numbers. Moreover, we also prove that $C_i(q)\in\mathbb{Q}[E_2,E_4,E_6]$, where E_2, E_4 and E_6 are the classical Eisenstein series of weight 2, 4 and 6, respectively.

© 2018 Published by Elsevier Inc.

* Corresponding author.
E-mail addresses: wanglq@whu.edu.cn, mathlqwang@163.com (L. Wang), czhang67@jhu.edu (C. Zhang).

https://doi.org/10.1016/j.aim.2018.05.006
0001-8708/© 2018 Published by Elsevier Inc.

1. Introduction

Consider the function

$$f(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!} q^{n(n-1)/2} \tag{1.1}$$

where $x, q \in \mathbb{C},\ |q| \leq 1$. The function $f(x)$ is entire and is called the "deformed exponential function" since it reduces to $\exp(x)$ when $q = 1$. It appears naturally and frequently in pure and applied mathematics. In combinatorics, the function $f(x)$ relates closely to the generating function for Tutte polynomials of the complete graph K_n [24], the enumeration of acyclic digraphs [18] and inversions of trees [15]. It also relates to the Whittaker and Goncharov constants [2] in complex analysis, and the partition function of one-site lattice gas with fugacity x and two-particle Boltzmann weight q in statistical mechanics [19]. Moreover, one can verify that this function is the unique solution to the functional differential equation

$$y'(x) = y(qx),\ y(0) = 1, \tag{1.2}$$

which is a special case of the "pantograph equation" [6]. For more detailed discussions on this function, we may refer to the notes from Alan Sokal's talks [20].

Surprisingly, many important properties of this function remain open, e.g., the distribution of its zeros. In 1952, Nassif [17] studied (on Littlewood's suggestion) the asymptotic behaviours and the zeros of the entire function

$$\sum_{n=0}^{\infty} e^{n^2\sqrt{2}\pi i} z^{2n}/n!,$$

which equals to $f(q^{\frac{1}{2}}z^2)$ with $q = e^{2\sqrt{2}\pi i}$. He used the fact that $\sqrt{2}$ has a periodic continued fraction expansion. Later, Littlewood [11,12] considered generalizations to Taylor series whose coefficients have smoothly varying moduli and arguments of the form $e^{n^2\alpha\pi i}$, where α is a quadratic irrationality. See also [4,10,13,23] for the studies on the behaviours of these functions. To our knowledge, for general complex number q satisfying $|q| \leq 1$, the distribution of the zeros of $f(x)$ has not been completely understood up to now. A theorem of Eremenko cited in [21] considered the case where q lies in any compact set of the open unit disk $\mathbb{D}$. There are relatively more works on the model case where $0 < q < 1$. In 1972, Morris et al. [5] used a theorem of Laguerre to show that $f(x)$ has infinitely many real zeros and these zeros are all negative and simple. They also proved that there is no other zero for the analytic extension (to the complex plane) of $f(x)$ by using the so-called multiplier sequence (a modest gap in their proof was filled by Iserles [8]). Therefore, when $0 < q < 1$, the zeros of $f(x)$ form one strictly decreasing sequence of negative numbers (x_k) $(k \geq 1)$. We remark that in some previous works (e.g., [9,22]),

Download English Version:

https://daneshyari.com/en/article/8904755

Download Persian Version:

https://daneshyari.com/article/8904755

Daneshyari.com