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Stable quotients and the holomorphic anomaly equation

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A B S T R A C T

We study the fundamental relationship between stable quotient invariants and the B-model for local $\mathbb{P}^2$ in all genera. Our main result is a direct geometric proof of the holomorphic anomaly equation in the precise form predicted by B-model physics. The method yields new holomorphic anomaly equations for an infinite class of twisted theories on projective spaces.

An example of such a twisted theory is the formal quintic defined by a hyperplane section of $\mathbb{P}^4$ in all genera via the Euler class of a complex. The formal quintic theory is found to satisfy the holomorphic anomaly equations conjectured for the true quintic theory. Therefore, the formal quintic theory and the true quintic theory should be related by transformations which respect the holomorphic anomaly equations.

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0. Introduction

0.1. GW/SQ

To the A-model Gromov–Witten theory of a Calabi–Yau 3-fold X is conjecturally associated the B-model theory of a mirror Calabi–Yau 3-fold Y. At the genus 0 level, much is known about the geometry underlying the B-model: Hodge theory, period integrals, and the linear sigma model. In higher genus, mathematical techniques have been less successful. In toric Calabi–Yau geometries, the topological recursion of Eynard and Orantin provides a link to the B-model [5,16]. Another mathematical approach to the B-model (without the toric hypothesis) has been proposed by Costello and Li [14] following paths suggested by string theory. A very different mathematical view of the geometry of B-model invariants is pursued here.

Let $X_5 \subset \mathbb{P}^4$ be a (nonsingular) quintic Calabi–Yau 3-fold. The moduli space of stable maps to the quintic of genus g and degree d,

$$\overline{M}_g(X_5, d) \subset \overline{M}_g(\mathbb{P}^4, d)\,,$$

has virtual dimension 0. The Gromov–Witten invariants,[1]

$$N^{\mathsf{GW}}_{g,d} = \langle 1 \rangle^{\mathsf{GW}}_{g,d} = \int_{[\overline{M}_g(X_5,d)]^{vir}} 1\,, \tag{1}$$

have been studied for more than 20 years, see [15,18,24] for an introduction to the subject.

The theory of stable quotients developed in [28] was partially inspired by the question of finding a geometric approach to a higher genus linear sigma model. The moduli space of stable quotients for the quintic,

$$\overline{Q}_g(X_5, d) \subset \overline{Q}_g(\mathbb{P}^4, d)\,,$$

was defined in [28, Section 9], and questions about the associated integral theory,

$$N^{\mathsf{SQ}}_{g,d} = \langle 1 \rangle^{\mathsf{SQ}}_{g,d} = \int_{[\overline{Q}_g(X_5,d)]^{vir}} 1\,, \tag{2}$$

were posed.

The existence of a natural obstruction theory on $\overline{Q}_g(X_5, d)$ and a virtual fundamental class $[\overline{Q}_g(X_5, d)]^{vir}$ is easily seen[2] in genus 0 and 1. A proposal in higher genus for the

[1] In degree 0, the moduli spaces of maps and stable quotients are both empty for genus 0 and 1, so the invariants in these cases vanish.

[2] For stability, marked points are required in genus 0 and positive degree is required in genus 1.