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Dichotomies, structure, and concentration in normed spaces

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ABSTRACT

We use probabilistic, topological and combinatorial methods to establish the following deviation inequality: For any normed space $X = (\mathbb{R}^n, \|\cdot\|)$ there exists an invertible linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ with

$$\begin{aligned}&\mathbb{P}\left(\big|\|TG\| - \mathbb{E}\|TG\|\big| > \varepsilon\mathbb{E}\|TG\|\right)\\ &\quad \leq C\exp\left(-c\max\{\varepsilon^2, \varepsilon\}\log n\right), \quad \varepsilon > 0,\end{aligned}$$

where G is the standard n-dimensional Gaussian vector and $C, c > 0$ are universal constants. It follows that for every $\varepsilon \in (0,1)$ and for every normed space $X = (\mathbb{R}^n, \|\cdot\|)$ there exists a k-dimensional subspace of X which is $(1+\varepsilon)$-Euclidean and $k \geq c\varepsilon \log n / \log \frac{1}{\varepsilon}$. This improves by a logarithmic on ε term the best previously known result due to G. Schechtman.

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1. Introduction

The concentration inequality in Gauss' space states that for any Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}$ with $|f(x) - f(y)| \leq L\|x - y\|_2$ for all $x, y \in \mathbb{R}^n$ one has

$$\mathbb{P}\left(|f(G) - \mathbb{E}f(G)| > t\right) \leq 2\exp(-\tfrac{1}{2}t^2/L^2), \quad t > 0, \tag{1.1}$$

where G is the standard n-dimensional Gaussian vector (for a proof the reader is referred to [43]; see [34] for the precise constants). This inequality is the prototype of what is called nowadays the concentration of measure phenomenon, one of the most important ideas in modern probability theory. This fundamental tool was put forward in the local theory of normed spaces in early 70's by V. Milman. Applying (1.1) for a norm $\|\cdot\|$ on $\mathbb{R}^n$ we get

$$\mathbb{P}\left(\big|\|G\| - \mathbb{E}\|G\|\big| > t\mathbb{E}\|G\|\right) \leq 2\exp(-\tfrac{1}{2}t^2 k), \quad t > 0, \tag{1.2}$$

where $k = k(X) = k(B_X) := (\mathbb{E}\|G\|/b)^2$ is referred to as the *critical dimension* (or *Dvoretzky number*) of the normed space $X = (\mathbb{R}^n, \|\cdot\|)$ and $b = b(X) = b(B_X)$ is the Lipschitz constant of the norm $\|\cdot\|$, i.e. $b = \max\{\|\theta\| : \|\theta\|_2 = 1\}$. It is well known that the above estimate is sharp in the large deviation regime, namely

$$\mathbb{P}(\|G\| \geq (1+t)\mathbb{E}\|G\|) \geq c\exp(-Ct^2 k), \quad t \geq 1, \tag{1.3}$$

where $c, C > 0$ are universal constants[3] (see e.g. [28, Corollary 3.2], [30, Statement 3.1] and [41, Proposition 2.10]). In the small deviation regime $0 < t < 1$ there exist many important examples which show that the obtained bounds are suboptimal; see [41] and [57] for a detailed discussion. Ideally one would like to know what properties of the underlying function improve the concentration. An example of such a result was recently obtained by the authors in [40] where they proved that a one-sided, variance-sensitive Gaussian small deviation inequality is valid for all convex functions.

This work is also concerned with optimal forms of the Gaussian concentration but the main focus will be on norms. Before stating the main problem of study, let us try to motivate the question which describes it. It is known (see e.g. [35], [43]) that for any norm $\|\cdot\|$ on $\mathbb{R}^n$, there exists a $T \in GL(n)$ such that

$$\mathbb{P}\left(\big|\|TG\| - \mathbb{E}\|TG\|\big| > t\mathbb{E}\|TG\|\right) \leq C\exp(-ct^2\log n), \quad t > 0, \tag{1.4}$$

where $G \sim N(\mathbf{0}, I_n)$. This follows from the fact that there exists a *position* (i.e. an invertible linear image) $T^{-1}(B_X)$ of the unit ball $B_X = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ for which the

[3] Here and everywhere else $C, c, C_1, c_1, \ldots$ stand for positive universal constants whose values may change from line to line. For any two quantities A, B depending on the dimension, on the parameters of the problem, etc. we write $A \simeq B$ if there exists a universal constant $C > 0$-independent of everything—such that $A \leq CB$ and $B \leq CA$.