Advances in Mathematics 332 (2018) 465–475

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Schubert polynomials as integer point transforms of generalized permutahedra ☆

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ARTICLE INFO

Article history:
Received 20 June 2017
Received in revised form 2 February 2018
Accepted 7 May 2018
Available online 26 May 2018
Communicated by Ezra Miller

Keywords:
Schubert polynomial
Key polynomial
Dual character of the flagged Weyl module
Newton polytope
Integer point transform
Generalized permutahedron

ABSTRACT

We show that the dual character of the flagged Weyl module of any diagram is a positively weighted integer point transform of a generalized permutahedron. In particular, Schubert and key polynomials are positively weighted integer point transforms of generalized permutahedra. This implies several recent conjectures of Monical, Tokcan and Yong.

☆ Fink is partially supported by an Engineering and Physical Sciences Research Council grant (EP/M01245X/1), and Mészáros by a National Science Foundation Grant (DMS 1501059).

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https://doi.org/10.1016/j.aim.2018.05.028
0001-8708/

1. Introduction

Schubert polynomials and key polynomials are classical objects in algebraic combinatorics. Schubert polynomials, introduced by Lascoux and Schützenberger in 1982 [13], represent cohomology classes of Schubert cycles in flag varieties. Key polynomials, also known as Demazure characters, are polynomials associated to compositions. Key polynomials were first introduced by Demazure for Weyl groups [6], and studied in the context of the symmetric group by Lascoux and Schützenberger in [14,15].

Schubert and key polynomials play an important role in algebraic combinatorics [2,3,9,11,21]. The second author and Escobar [8] showed that for permutations $1\pi'$ where π' is dominant, Schubert polynomials are specializations of reduced forms in the subdivision algebra of flow and root polytopes. On the other hand, intimate connections of flow and root polytopes with generalized permutahedra have been exhibited by Postnikov [20], and more recently by the last two authors [18]. These works imply that for permutations $1\pi'$ where π' is dominant, the Schubert polynomial $\mathfrak{S}_{1\pi'}(\mathbf{x})$ is equal to the integer point transform of a generalized permutahedron [18].

Using realizations of Schubert polynomials and key polynomials as dual characters of a certain module [16,21], the main result of this paper proves that the Newton polytope of any Schubert or key polynomial is a generalized permutahedron, and that any such polynomial equals a sum over the lattice points of its Newton polytope with positive integral coefficients.

After reviewing the necessary background, we prove our main theorem and draw corollaries about Schubert and key polynomials, confirming several recent conjectures of Monical, Tokcan and Yong [19].

2. Background

This section contains a collection of definitions and facts relevant to the main result of this paper. Our basic notions are Schubert polynomials, key polynomials, Newton polytopes, generalized permutahedra, (Schubert) matroids, and flagged Weyl modules.

2.1. Schubert polynomials

The Schubert polynomial of the longest permutation $w_0 = n\; n{-}1\; \cdots\; 2\; 1 \in S_n$ is

$$\mathfrak{S}_{w_0} := x_1^{n-1}x_2^{n-2}\cdots x_{n-1}.$$

For $w \in S_n$, $w \neq w_0$, there exists $i \in [n-1]$ such that $w(i) < w(i+1)$. For any such i, the **Schubert polynomial** $\mathfrak{S}_w$ ([13]) is defined as

$$\mathfrak{S}_w(x_1, \dots, x_n) := \partial_i \mathfrak{S}_{ws_i}(x_1, \dots, x_n),$$

where ∂_i is the ith divided difference operator