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Rigidity of inversive distance circle packings revisited

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A B S T R A C T

Inversive distance circle packing metric was introduced by P Bowers and K Stephenson [7] as a generalization of Thurston's circle packing metric [34]. They conjectured that the inversive distance circle packings are rigid. For nonnegative inversive distance, Guo [22] proved the infinitesimal rigidity and then Luo [27] proved the global rigidity. In this paper, based on an observation of Zhou [37], we prove this conjecture for inversive distance in $(-1, +\infty)$ by variational principles. We also study the global rigidity of a combinatorial curvature introduced in [14,16,19] with respect to the inversive distance circle packing metrics where the inversive distance is in $(-1, +\infty)$.

1. Introduction

1.1. Background

In his work on constructing hyperbolic structure on 3-manifolds, Thurston ([34], Chapter 13) introduced the notion of circle packing metric on triangulated surfaces with

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non-obtuse intersection angles. The requirement of prescribed intersection angles corresponds to the fact that the intersection angle of two circles is invariant under the Möbius transformations. For triangulated surfaces with Thurston's circle packing metrics, there are singularities at the vertices. The classical combinatorial curvature K_i is introduced to describe the singularity at the vertex v_i, which is defined as the angle deficit at v_i. Thurston's work generalized Andreev's work on circle packing metrics on a sphere [1,2] and gave a complete characterization of the space of the classical combinatorial curvature. As a corollary, he obtained the combinatorial-topological obstacle for the existence of a constant curvature circle packing with non-obtuse intersection angles, which could be written as combinatorial-topological inequalities. Zhou [37] recently generalized Andreev–Thurston Theorem to the case that the intersection angles are in $[0, \pi)$. Chow and Luo [9] introduced a combinatorial Ricci flow, a combinatorial analogue of the smooth surface Ricci flow, for triangulated surfaces with Thurston's circle packing metrics and established the equivalence between the existence of a constant curvature circle packing metric and the convergence of the combinatorial Ricci flow.

Inversive distance circle packing on triangulated surfaces was introduced by Bowers and Stephenson [7] as a generalization of Thurston's circle packing. Different from Thurston's circle packing, adjacent circles in inversive distance circle packing are allowed to be disjoint and the relative distance of the adjacent circles is measured by the inversive distance, which is a generalization of intersection angle. See Bowers–Hurdal [6], Stephenson [33] and Guo [22] for more information. The inversive distance circle packings have practical applications in medical imaging and computer graphics, see [24,35,36] for example. Bowers and Stephenson [7] conjectured that the inversive distance circle packings are rigid. Guo [22] proved the infinitesimal rigidity and then Luo [27] solved affirmably the conjecture for nonnegative inversive distance with Euclidean and hyperbolic background geometry. For the spherical background geometry, Ma and Schlenker [29] had a counterexample showing that there is in general no rigidity and John C. Bowers and Philip L. Bowers [4] obtained a new construction of their counterexample using the inversive geometry of the 2-sphere. John Bowers, Philip Bowers and Kevin Pratt [5] recently proved the global rigidity of convex inversive distance circle packings in the Riemann sphere. Ge and Jiang [12,13] recently studied the deformation of combinatorial curvature and found a way to search for inversive distance circle packing metrics with constant cone angles. They also obtained some results on the image of curvature map for inversive distance circle packings. Ge and Jiang [14] and Ge and the author [19] further extended a combinatorial curvature introduced by Ge and the author in [16–18] to inversive distance circle packings and studied the rigidity and deformation of the curvature.

In this paper, based on an obversion of Zhou [37], we prove Bowers and Stephenson's rigidity conjecture for inversive distance in $(-1, +\infty)$. The main tools are the variational principle established by Guo [22] for inversive distance circle packings and the extension of locally convex function introduced by Bobenko, Pinkall and Springborn [3] and systematically developed by Luo [27]. We refer to Glickenstein [20] for a nice geomet-