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Non-classification of Cartan subalgebras for a class of von Neumann algebras

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ABSTRACT

We study the complexity of the classification problem for Cartan subalgebras in von Neumann algebras. We construct a large family of II_1 factors whose Cartan subalgebras up to unitary conjugacy are not classifiable by countable structures, providing the first such examples. Additionally, we construct examples of II_1 factors whose Cartan subalgebras up to conjugacy by an automorphism are not classifiable by countable structures. Finally, we show directly that the Cartan subalgebras of the hyperfinite II_1 factor up to unitary conjugacy are not classifiable by countable structures, and deduce that the same holds for any McDuff II_1 factor with at least one Cartan subalgebra.

0. Introduction

Since the first paper of Murray and von Neumann [27] the study of von Neumann algebras has been closely related to the study of group actions on measure spaces. The *group measure space construction* introduced in their paper associates to every free er-

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godic probability measure preserving (pmp) action $\Gamma \curvearrowright (X, \mu)$ of a countable group Γ on a probability measure space (X, μ) a crossed product von Neumann algebra $L^\infty(X) \rtimes \Gamma$, which turns out to be a factor of type II_1. The classification of these group measure space von Neumann algebras is in general a very hard problem. Nevertheless a lot of progress has been made on several fronts.

One recurring theme in all known classification results is the special role played by the so-called Cartan subalgebras – maximal abelian subalgebras whose normalizer generates the II_1 factor. Given a free ergodic pmp action $\Gamma \curvearrowright X$, the group measure space II_1 factor $L^\infty(X) \rtimes \Gamma$ always contains $L^\infty(X)$ as a Cartan subalgebra. Moreover, Singer established in [40] that two free ergodic pmp actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are *orbit equivalent* if and only if there exists an isomorphism $\theta : L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ such that $\theta(L^\infty(X)) = L^\infty(Y)$. This result both gives an immediate link with measured group theory and highlights the importance of understanding the Cartan subalgebras of a given II_1 factor. For instance, if one can show that certain classes of group measure space II_1 factors have a unique Cartan subalgebra (up to conjugacy by an automorphism), their classification up to isomorphism reduces to the classification of the corresponding actions up to orbit equivalence.

If the acting group is amenable the group measure space construction will always give rise to the hyperfinite II_1 factor R by Connes' famous theorem [3]. Moreover it was shown in [6] that all Cartan subalgebras of R are conjugate by an automorphism of R. Hence all free ergodic pmp actions of all amenable groups are orbit equivalent and besides the group being amenable, one cannot recover any more information about the group nor the action from the group measure space II_1 factor.

On the other hand, for non-amenable groups there is a wide spectrum of different II_1 factors appearing and rigidity results start showing up. During the last 15 years, Popa's *deformation/rigidity* theory has led to considerable progress in the classification of group measure space II_1 factors, see for instance the surveys [35,46,16]. In particular, several uniqueness results for Cartan subalgebras have been established. The first such result was obtained by Ozawa and Popa in [30]. They proved that $L^\infty(X)$ is the unique Cartan subalgebra up to unitary conjugacy inside $L^\infty(X) \rtimes \mathbb{F}_n$ where $\mathbb{F}_n$ is a non-abelian free group and $\mathbb{F}_n \curvearrowright X$ is a free ergodic *profinite* pmp action. The class of groups whose profinite actions give II_1 factors with a unique Cartan subalgebra was subsequently extended in [31,1]. Later the condition of profiniteness was removed by Popa and Vaes, who showed in [37] that *any* free ergodic pmp action of a non-abelian free group gives rise to a II_1 factor with a unique Cartan subalgebra up to unitary conjugacy. Hereafter, additional uniqueness results were obtained in (among others) [38,17,2].

Of course not every II_1 factor has a unique Cartan subalgebra. The first II_1 factor M containing at least two Cartan subalgebras that are not conjugate by an automorphism of M was constructed in [4]. By now there are more examples known (see for instance [31,36]). Nevertheless it is worth mentioning that at the moment, as soon as uniqueness fails, we do not have any examples of II_1 factors for which we can describe all Cartan subalgebras up to unitary conjugacy in a satisfactory way. Some progress in this direction