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Convex cones, integral zonotopes, limit shape

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ABSTRACT

Given a convex cone C in $\mathbb{R}^d$, an integral zonotope T is the sum of segments $[0, \mathbf{v}_i]$ $(i = 1, \ldots, m)$ where each $\mathbf{v}_i \in C$ is a vector with integer coordinates. The endpoint of T is $\mathbf{k} = \sum_1^m \mathbf{v}_i$. Let $\mathcal{T}(C, \mathbf{k})$ be the family of all integral zonotopes in C whose endpoint is $\mathbf{k} \in C$. We prove that, for large $\mathbf{k}$, the zonotopes in $\mathcal{T}(C, \mathbf{k})$ have a limit shape, meaning that, after suitable scaling, the overwhelming majority of the zonotopes in $\mathcal{T}(C, \mathbf{k})$ are very close to a fixed convex set. We also establish several combinatorial properties of a typical zonotope in $\mathcal{T}(C, \mathbf{k})$.

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1. Introduction and main results

This paper is about convex cones C in $\mathbb{R}^d$, integral zonotopes contained in C, and their limit shape. The cone C is going to be closed, convex and pointed (that is no line lies in C) and its interior, $\operatorname{Int} C$, is non-empty. We write $\mathcal{C}$ or $\mathcal{C}^d$ for the set of these cones.

A convex (lattice) polytope $T \subset C$ is an integral zonotope if there exists $m \in \mathbb{N}$ and $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{Z}^d \cap C$ (that is, each $\mathbf{v}_i$ is lattice point in C) such that

$$T = \left\{ \sum_{i=1}^{m} \alpha_i \mathbf{v}_i \mid (\alpha_1, \dots, \alpha_m) \in [0,1]^m \right\}$$
$$= \operatorname{Conv} \left\{ \sum_{i=1}^{m} \varepsilon_i \mathbf{v}_i \mid (\varepsilon_1, \dots, \varepsilon_m) \in \{0,1\}^m \right\}.$$

The multiset $V = \{\mathbf{v}_1, \dots, \mathbf{v}_m\} \subset \mathbb{Z}^d$ determines $T = T(V)$ uniquely, of course, but not conversely. More about this later. The endpoint of T is just $\sum_{i=1}^{m} \mathbf{v}_i$. Define $\mathcal{T}(C, \mathbf{k})$ as the family of all integral zonotopes in C whose endpoint is $\mathbf{k} \in \mathbb{Z}^d \cap \operatorname{Int} C$. Clearly, $\mathcal{T}(C, \mathbf{k})$ is a finite set. Let $p(C, \mathbf{k})$ denote its cardinality.

The main result of this paper is that, for large $\mathbf{k}$, the overwhelming majority of the elements of $\mathcal{T}(C, \mathbf{k})$ are very close to a fixed convex set $T_0 = T_0(C, \mathbf{k})$ which is actually a zonoid. We write $\operatorname{dist}(A, B)$ for the Hausdorff distance of the sets $A, B \subset \mathbb{R}^d$. Here comes our main result.

Theorem 1.1. *Given $C \in \mathcal{C}^d$ $(d \geq 2)$ and $\mathbf{k} \in \operatorname{Int} C$ there is a convex set $T_0 = T_0(C, \mathbf{k})$ such that for every $\varepsilon > 0$,*

$$\lim_{n \to \infty} \frac{\operatorname{card} \left\{ T \in \mathcal{T}(C, n\mathbf{k}) \mid \operatorname{dist}(\frac{1}{n} T, T_0) > \varepsilon \right\}}{p(C, n\mathbf{k})} = 0.$$

This result has been known for $d = 2$. Twenty years ago, Bárány [1], Sinai [13] and Vershik [15] proved the existence of a limit shape for the set of all convex lattice polygons lying in the square $[-n, n]^2$ endowed with the uniform distribution. Although not all convex lattice polygons are (translates of) zonotopes, case $d = 2$ of Theorem 1.1 follows directly from their result. The approach of these papers relies on a natural link between convex lattice polygons on the first hand, and integer partitions on the other hand.

In addition to Theorem 1.1, the asymptotic behavior as $n \to \infty$ of $p(C, n\mathbf{k})$ can also be determined.

Theorem 1.2. *Under the above conditions on C and $\mathbf{k}$ there is a number $q(C, \mathbf{k}) > 0$ such that, as n tends to infinity,*

$$n^{-\frac{d}{d+1}} \log p(C, n\mathbf{k}) \longrightarrow c_d \, q(C, \mathbf{k}),$$

where $c_d = \sqrt[d+1]{\frac{\zeta(d+1)}{\zeta(d)} (d+1)!}$ depends only on the dimension.