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Residue current approach to Ehrenpreis–Malgrange type theorem for linear differential equations with constant coefficients and commensurate time lags

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A B S T R A C T

We introduce a ring $\mathcal{H}$ of partial difference-differential operators with constant coefficients initially defined by H. Glüsing-Lürßen for ordinary difference-differential operators and investigate its cohomological properties. Combining this ring theoretic observation with the integral representation technique developed by M. Andersson, we solve a certain type of division with bounds. In the last section, we deduce from this injectivity properties of various function modules over $\mathcal{H}$ as well as the density results of exponential polynomial solutions for partial difference-differential equations.

1. Introduction

In the last century, various studies on the general theory of linear partial differential equations with constant coefficients has been conducted. Among many important works, we should mention the celebrated work of L. Ehrenpreis: Ehrenpreis' fundamental principle. Roughly speaking, Ehrenpreis' fundamental principle means that any solution of a system of linear partial differential equations with constant coefficients

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can be written as a superposition of exponential polynomial solutions. In particular, exponential polynomial solutions are dense in the solution space.[1] Furthermore, if we let $\mathbb{C}[\partial] = \mathbb{C}[\partial_1, \cdots, \partial_n]$ denote the ring of linear partial differential operators with constant coefficients, Ehrenpreis fundamental principle has its cohomological counterpart which is expressed as injectivity properties of function spaces over $\mathbb{C}[\partial]$. We give a statement of a fundamental result in the theory of linear partial differential equations only for C^∞ functions.

Theorem 1.1 *([13, Theorem 4.2], [17, Theorem 7.6.12, 7.6.13], [27, IV,§5,5°]). Let Ω be a convex subset of $\mathbb{R}^n$ and let $\mathbf{E}$ denote the $\mathbb{C}[\partial]$-module consisting of exponential polynomials. One has the following properties:*

(1) *$C^\infty(\Omega)$ is an injective $\mathbb{C}[\partial]$-module, i.e., for any finitely generated $\mathbb{C}[\partial]$-module M and any positive integer i, one has an identity*

$$\mathrm{Ext}^i_{\mathbb{C}[\partial]}(M, C^\infty(\Omega)) = 0. \tag{1.1}$$

(2) *For any matrix $\mathbb{P}(\partial) \in M(r_1, r_0; \mathbb{C}[\partial])$, $\mathrm{Ker}\left(\mathbb{P}(\partial) : \mathbf{E}^{r_0\times 1} \to \mathbf{E}^{r_1\times 1}\right)$ is dense in $\mathrm{Ker}(\mathbb{P}(\partial) : C^\infty(\Omega)^{r_0\times 1} \to C^\infty(\Omega)^{r_1\times 1})$.*

We call the theorem above Ehrenpreis–Malgrange type theorem. In the following, we explain the meaning of (1) of Theorem 1.1. To this end, let us consider a more general setting.

Let R be a subring of the ring of distributions with compact supports $\mathcal{E}'(\mathbb{R}^n)$ and F be a vector space of functions of a suitable kind, say $F = C^\infty(\mathbb{R}^n)$ for simplicity. Then, R acts on F by

$$T \cdot f \overset{def}{=} T * f \quad (T \in R, f \in F). \tag{1.2}$$

Here, $*$ is the usual convolution product. Note that usual partial differential operator $\frac{\partial}{\partial x_i}$ can be realized as a convolution operator by

$$\frac{\partial f}{\partial x_i} = \left(\frac{\partial}{\partial x_i}\delta\right) * f, \tag{1.3}$$

where δ is Dirac measure with support at the origin. For any $r_1 \times r_0$ matrix $\mathbb{P}$ with entries in R and for any $\mathbf{f} = {}^t(f_1, \cdots, f_{r_1}) \in F^{r_1\times 1}$, we consider a system of equations

$$\mathbb{P}\mathbf{u} = \mathbf{f}, \tag{1.4}$$

[1] This type of theorem can never be true in variable coefficients cases. In this paper, "difference-differential equations" always means linear difference-differential equations with constant coefficients.