Advances in Mathematics 331 (2018) 209–238

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

The Hopf algebra of skew shapes, torsion sheaves on $\mathbb{A}^n_{/\mathbb{F}_1}$, and ideals in Hall algebras of monoid representations

Matt Szczesny

Department of Mathematics and Statistics, Boston University, 111 Cummington Mall, Boston, United States

ARTICLE INFO

Article history:
Received 10 July 2017
Received in revised form 15 March 2018
Accepted 17 March 2018
Available online 24 April 2018
Communicated by Tony Pantev

Keywords:
Combinatorial Hopf algebra
Field of one element
Skew shape
Lie algebra
Monoid scheme
Torsion sheaf

ABSTRACT

We study ideals in Hall algebras of monoid representations on pointed sets corresponding to certain conditions on the representations. These conditions include the property that the monoid act via partial permutations, that the representation possess a compatible grading, and conditions on the support of the module. Quotients by these ideals lead to combinatorial Hopf algebras which can be interpreted as Hall algebras of certain sub-categories of modules. In the case of the free commutative monoid on n generators, we obtain a co-commutative Hopf algebra structure on n-dimensional skew shapes, whose underlying associative product amounts to a "stacking" operation on the skew shapes. The primitive elements of this Hopf algebra correspond to connected skew shapes, and form a graded Lie algebra by anti-symmetrizing the associative product. We interpret this Hopf algebra as the Hall algebra of a certain category of coherent torsion sheaves on $\mathbb{A}^n_{/\mathbb{F}_1}$ supported at the origin, where $\mathbb{F}_1$ denotes the field of one element. This Hopf algebra may be viewed as an n-dimensional generalization of the Hopf algebra of symmetric functions, which corresponds to the case $n = 1$.

E-mail address: szczesny@math.bu.edu.

https://doi.org/10.1016/j.aim.2018.03.032
0001-8708/

1. Introduction

This paper introduces a Lie algebra structure $\mathfrak{sk}_n$ on n-dimensional connected skew shapes. The enveloping algebra $\mathbb{U}(\mathfrak{sk}_n)$ is constructed as the Hall algebra of a subcategory of torsion sheaves on $\mathbb{A}^n_{/\mathbb{F}_1}$ supported at the origin, where $\mathbb{F}_1$ denotes the "field of one element". The connected skew shapes correspond to the indecomposable objects of this category. In the introduction below we explain how the Hall algebra construction, applied in non-additive contexts such as that of algebraic geometry over $\mathbb{F}_1$ produces combinatorial Hopf algebras of a representation theoretic nature which can be viewed as degenerations of quantum-group like objects over $\mathbb{F}_q$.

1.1. Hall algebras of Abelian categories

The study of Hall algebras is by now a well-established area with several applications in representation theory and algebraic geometry (see [15] for a very nice overview). We briefly recall the generic features of the most basic version of this construction. Given an abelian category $\mathcal{C}$, let

$$Fl_i(\mathcal{C}) := \{A_0 \subset A_1 \subset \cdots \subset A_i |\ A_k \in \mathrm{Ob}(\mathcal{C})\}$$

denote the stack parametrizing flags of objects in $\mathcal{C}$ of length $i+1$ (viewed here simply as a set). Thus

$$Fl_0(\mathcal{C}) = \mathrm{Iso}(\mathcal{C}),$$

the moduli stack of isomorphism classes of objects of $\mathcal{C}$, and

$$Fl_1(\mathcal{C}) = \{A_0 \subset A_1 |\ A_0, A_1 \in \mathrm{Ob}(\mathcal{C})\}$$

is the usual Hecke correspondence. We have maps

$$\pi_i : Fl_1(\mathcal{C}) \to Fl_0(\mathcal{C}),\ i = 1, 2, 3 \tag{1}$$

where

$$\begin{aligned} \pi_1(A_0 \subset A_1) &= A_0 \\ \pi_2(A_0 \subset A_1) &= A_1 \\ \pi_3(A_0 \subset A_1) &= A_1/A_0 \end{aligned}$$

We may then attempt to define the Hall algebra of $\mathcal{C}$ as the space of $\mathbb{Q}$-valued functions on $Fl_0(\mathcal{C})$ with finite support, i.e.

$$\mathrm{H}_{\mathcal{C}} = \mathbb{Q}_c[Fl_0(\mathcal{C})]$$