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On the embeddability of real hypersurfaces into hyperquadrics

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ABSTRACT

A well known result of Forstnerič [15] states that most real-analytic strictly pseudoconvex hypersurfaces in complex space are not holomorphically embeddable into spheres of higher dimension. A more recent result by Forstnerič [16] states even more: most real-analytic hypersurfaces do not admit a holomorphic embedding even into a merely algebraic hypersurface of higher dimension, in particular, a hyperquadric. We emphasize that both cited theorems are proved by showing that the set of embeddable hypersurfaces is a set of first Baire category. At the same time, the classical theorem of Webster [30] asserts that *every* real-algebraic Levi-nondegenerate hypersurface admits a transverse holomorphic embedding into a nondegenerate real hyperquadric in complex space.

In this paper, we provide *effective* results on the non-embeddability of real-analytic hypersurfaces into a hyperquadric. We show that, under the codimension restriction $N \leq 2n$, the defining functions $\varphi(z, \bar{z}, u)$ of all real-analytic hypersurfaces $M = \{v = \varphi(z, \bar{z}, u)\} \subset \mathbb{C}^{n+1}$ containing Levi-nondegenerate points and locally transversally holomorphically embeddable into some hyperquadric $\mathcal{Q} \subset \mathbb{C}^{N+1}$ satisfy an *universal* algebraic partial differential equation $D(\varphi) = 0$, where the algebraic-differential operator $D = D(n, N)$ depends on $n \geq 1$, $n < N \leq 2n$ only. To the best of

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our knowledge, this is the first effective result characterizing real-analytic hypersurfaces embeddable into a hyperquadric of higher dimension. As an application, we show that for every n, N as above there exists $\mu = \mu(n, N)$ such that a Zariski generic real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$ of degree $\geq\mu$ is not transversally holomorphically embeddable into any hyperquadric $\mathcal{Q} \subset \mathbb{C}^{N+1}$. We also provide an explicit upper bound for μ in terms of n, N. To the best of our knowledge, this gives the first effective lower bound for the CR-complexity of a Zariski generic real-algebraic hypersurface in complex space of a fixed degree.

1. Introduction

Let $M \subset \mathbb{C}^{n+1}$, $n \geq 1$ be a real-analytic Levi-nondegenerate hypersurface. The celebrated theory due to Chern and Moser [6] (see also Cartan [5]) asserts that only a very rare such M admits a local biholomorphic mapping into a nondegenerate real hyperquadric

$$\mathcal{Q} = \left\{[\xi_0, ..., \xi_{n+1}] \in \mathbb{CP}^{n+1} : \ |\xi_0|^2 + ... + |\xi_k|^2 - |\xi_{k+1}|^2 - ... - |\xi_{n+1}|^2 = 0\right\}.$$

Moreover, Chern and Moser show that the existence of the desired biholomorphic mapping into a hyperquadric is equivalent to vanishing of a special *CR-curvature* of a real hypersurface M.

A natural problem to pursue in view of the Chern–Moser theory is the possibility to construct a local holomorphic embedding $F : (M, p) \mapsto (\mathcal{Q}, p')$ of a real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$, $n \geq 1$ into a hyperquadric $\mathcal{Q} \subset \mathbb{C}^{N+1}$ *of higher dimension*. Here by a holomorphic embedding F of $M \subset \mathbb{C}^n$ into $M' \subset \mathbb{C}^N$, we mean a holomorphic embedding of an open neighborhood U of M in $\mathbb{C}^n$ into a neighborhood U' of M' in $\mathbb{C}^N$, sending M into M'. One usually presumes certain nondegeneracy conditions for the mapping F, such as *transversality* (the latter means that $dF(\mathbb{C}^{n+1})|_p \not\subset T_{p'}\mathcal{Q}$).

The existence of a transversal holomorphic embedding into a hyperquadric can be viewed as a *finite CR-complexity* of a real hypersurface (see, e.g., [13]). The latter is addressed as the minimal possible difference $N - n$ between the CR-dimensions of the target hyperquadric and the source real hypersurface. An alternative approach to complexity in CR-geometry is due to the school of D'Angelo, see, e.g., [7–9]. A strong motivation for studying the embedding problem is the celebrated theorem of Webster [30] which states that *every* real-algebraic Levi-nondegenerate hypersurface admits a transverse holomorphic embedding into a nondegenerate real hyperquadric in complex space. Thus, every algebraic Levi-nondegenerate hypersurface has a finite CR-complexity.

Since the work of Webster, a large number of publications have been dedicated to studying holomorphic embeddings of real hypersurfaces into hyperquadrics. However,