Advances in Mathematics 331 (2018) 268–325

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs ☆

Marcelo Laca [a], Iain Raeburn [b], Jacqui Ramagge [c,*], Michael F. Whittaker [d]

[a] *Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3P4, Canada*
[b] *School of Mathematics and Statistics, Victoria University of Wellington, PO Box 600, Wellington 6140, New Zealand*
[c] *School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia*
[d] *School of Mathematics and Statistics, University of Glasgow, University Place, Glasgow G12 8QQ, United Kingdom*

ARTICLE INFO

Article history:
Received 4 October 2016
Accepted 17 March 2018
Available online 25 April 2018
Communicated by Dan Voiculescu

Keywords:
States
Operator algebras
Self-similar actions
Groupoids

ABSTRACT

We consider self-similar actions of groupoids on the path spaces of finite directed graphs, and construct examples of such self-similar actions using a suitable notion of graph automaton. Self-similar groupoid actions have a Cuntz–Pimsner algebra and a Toeplitz algebra, both of which carry natural dynamics lifted from the gauge actions. We study the equilibrium states (the KMS states) on the resulting dynamical systems. Above a critical inverse temperature, the KMS states on the Toeplitz algebra are parametrised by the traces on the full C^*-algebra of the groupoid, and we describe a program for finding such traces. The critical inverse temperature is the logarithm of the spectral radius of the incidence matrix of the graph, and at the critical temperature the KMS states on the Toeplitz algebra factor through states of the Cuntz–Pimsner algebra. Under a verifiable hypothesis on the self-similar action, there is a unique KMS state on the Cuntz–Pimsner algebra. We discuss an explicit method of

☆ This research was supported by the Natural Sciences and Engineering Research Council of Canada, the Marsden Fund of the Royal Society of New Zealand, and the Australian Research Council.
* Corresponding author.
E-mail addresses: laca@math.uvic.ca (M. Laca), iain.raeburn@vuw.ac.nz (I. Raeburn), Jacqui.Ramagge@sydney.edu.au (J. Ramagge), Mike.Whittaker@glasgow.ac.uk (M.F. Whittaker).

https://doi.org/10.1016/j.aim.2018.03.030
0001-8708/

computing the values of this KMS state, and illustrate with examples.

1. Introduction

A self-similar group (G, X) consists of a finite set X and a faithful action of a group G on the set X^* of finite words in X, such that: for each $g \in G$ and $x \in X$, there exists $h \in G$ satisfying

$$g \cdot (xw) = (g \cdot x)(h \cdot w) \text{ for all } w \in X^*.$$

Self-similar groups are often defined recursively using data presented in an automaton (see, for example, [20, Chapter 1] or §2 below). To each self-similar group (G, X), Nekrashevych associated a C^*-algebra $\mathcal{O}(G, X)$, which is by definition the Cuntz–Pimsner algebra of a Hilbert bimodule over the reduced group algebra $C_r^*(G)$ [19,21].

We recently studied the Toeplitz algebra $\mathcal{T}(G, X)$ of this Hilbert bimodule [17]. Both $\mathcal{T}(G, X)$ and $\mathcal{O}(G, X)$ carry natural gauge actions of the unit circle, and composing with the exponential map gives actions α of the real line. In [17], we classified the equilibrium states (the KMS states) of the dynamical systems $(\mathcal{T}(G, X), \mathbb{R}, \alpha)$ and $(\mathcal{O}(G, X), \mathbb{R}, \alpha)$. We found a simplex of KMS states on $\mathcal{T}(G, X)$ at all inverse temperatures larger than a critical value $\ln |X|$, and showed, under a mild hypothesis on (G, X), that there is a single KMS state on $\mathcal{O}(G, X)$ whose inverse temperature is $\ln |X|$.

Here we consider a new kind of self-similarity involving an action of a groupoid G on the path space E^* of a finite directed graph E, which we view as a forest of trees $\{vE^* : v \in E^0\}$. An equation of the form

$$g \cdot (e\mu) = (g \cdot e)(h \cdot \mu)$$

for paths $e\mu$ defines an isomorphism h of the subtree $s(e)E^*$ onto $s(g \cdot e)E^*$. We call h a *partial isomorphism* of E^*. The partial isomorphisms of E^* form a groupoid $\mathrm{PIso}(E^*)$ with unit space E^0. A self-similar action of a groupoid G with unit space E^0 is then a groupoid homomorphism of G into $\mathrm{PIso}(E^*)$.

Our results are motivated by a construction of Exel and Pardo [8], who studied a family of self-similar actions of groups on path spaces. Their main motivation was to provide a unified theory that accommodates both Nekrashevych's Cuntz–Pimsner algebras and a family of "Katsura algebras" [13] that includes all Kirchberg algebras. We seek a common setting for the analyses of KMS states on self-similar groups in [17] and on the Toeplitz–Cuntz–Krieger algebras of graphs [7,12,10,11].

To each of our self-similar actions (G, E) we associate a Toeplitz algebra $\mathcal{T}(G, E)$ and a Cuntz–Pimsner algebra $\mathcal{O}(G, E)$, and study the dynamics arising from the gauge