Advances in Mathematics 331 (2018) 339–386

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

The equality cases of the Ehrhard–Borell inequality

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A R T I C L E I N F O

Article history:
Received 19 September 2017
Received in revised form 28 March 2018
Accepted 16 April 2018
Communicated by Gilles Pisier

MSC:
60G15
39B62
52A40
35K65
35B50

Keywords:
Ehrhard–Borell inequality
Equality cases
Gaussian measures
Gaussian Brunn–Minkowski inequalities
Degenerate parabolic equations

A B S T R A C T

The Ehrhard–Borell inequality is a far-reaching refinement of the classical Brunn–Minkowski inequality that captures the sharp convexity and isoperimetric properties of Gaussian measures. Unlike in the classical Brunn–Minkowski theory, the equality cases in this inequality are far from evident from the known proofs. The equality cases are settled systematically in this paper. An essential ingredient of the proofs are the geometric and probabilistic properties of certain degenerate parabolic equations. The method developed here serves as a model for the investigation of equality cases in a broader class of geometric inequalities that are obtained by means of a maximum principle.

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https://doi.org/10.1016/j.aim.2018.04.013

1. Introduction

The Brunn–Minkowski inequality plays a central role in numerous problems in different areas of mathematics [21,5,45]. In its simplest form, it states that

$$\mathrm{Vol}(\lambda A + \mu B)^{1/n} \geq \lambda \mathrm{Vol}(A)^{1/n} + \mu \mathrm{Vol}(B)^{1/n}$$

for nonempty closed sets $A, B \subseteq \mathbb{R}^n$ and $\lambda, \mu > 0$. In the nontrivial case $0 < \mathrm{Vol}(A), \mathrm{Vol}(B) < \infty$, equality holds precisely when A, B are homothetic and convex.

From the very beginning, the study of the cases of equality in the Brunn–Minkowski inequality has formed an integral part of the theory. Minkowski, who proved the inequality for convex sets, established the equality cases in this setting by a careful analysis of the proof [38, p. 247, (6)]. Both the inequality and equality cases were later extended to measurable sets [37,23]; the equality cases are first shown to reduce to the convex case, for which Minkowski's result can be invoked. The understanding of equality cases plays an important role in its own right. It provides valuable insight into the Brunn–Minkowski theory and into closely related inequalities, such as the Riesz–Sobolev inequality [12]. It also guarantees uniqueness of solutions to variational problems that arise in geometry and mathematical physics (including the classical isoperimetric problem), e.g., [21, §6] or [14, §4.1].

This paper is concerned with analogues of the Brunn–Minkowski inequality for the standard Gaussian measure γ_n on $\mathbb{R}^n$, which play an important role in probability theory. The Brunn–Minkowski inequality is rather special to the Lebesgue measure: while a weak form of the inequality holds for any log-concave measure [7], this property is not sufficiently strong to explain, for example, the isoperimetric properties of such measures. It is therefore remarkable that a sharp analogue of the Brunn–Minkowski inequality proves to exist for Gaussian measures. A simple form of this inequality is as follows; in the sequel, we denote $\Phi(x) := \gamma_1((-\infty, x])$.

Theorem 1.1 *(Ehrhard, Borell). For closed sets $A, B \subseteq \mathbb{R}^n$ and any $\lambda, \mu > 0$ such that $\lambda + \mu \geq 1$ and $|\lambda - \mu| \leq 1$, we have*

$$\Phi^{-1}(\gamma_n(\lambda A + \mu B)) \geq \lambda \Phi^{-1}(\gamma_n(A)) + \mu \Phi^{-1}(\gamma_n(B)).$$

If A, B are also convex, the conclusion remains valid assuming only $\lambda + \mu \geq 1$.

The Ehrhard–Borell inequality lies at the top of a large hierarchy of Gaussian inequalities. It implies the Gaussian isoperimetric inequality, which states that half-spaces minimize Gaussian surface area among all sets of the same measure; the Gaussian isoperimetric inequality in turn implies numerous geometric and analytic inequalities for Gaussian measures [35,33]. It has recently been understood that Theorem 1.1 gives rise to new concentration phenomena for convex functions that go beyond the isoperi-