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Gravity formality

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We show that Willwacher's cyclic formality theorem can be extended to preserve natural Gravity operations on cyclic multivector fields and cyclic multidifferential operators. We express this in terms of a homotopy Gravity quasi-isomorphism with explicit local formulas. For this, we develop operadic tools related to mixed complexes and cyclic homology and prove that the operad $\mathsf{M}_{\circlearrowleft}$ of natural operations on cyclic operators is formal and hence quasi-isomorphic to the Gravity operad.

Contents

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0. Introduction

The original deformation quantization problem aims to obtain a formal deformation of the associative product of functions of a Poisson manifold M, called a *star product*. The space governing such deformations is essentially the Lie algebra of multidifferential operators D_{poly}, the smooth version of the Hochschild complex of the algebra $C^\infty(M)$. In his celebrated paper [21], Kontsevich showed that the Lie algebra D_{poly} is formal i.e., it is quasi-isomorphic to its homology, the Lie algebra of multivector fields T_{poly}. His proof involves the construction of the "formality morphism", a homotopy quasi-isomorphism of Lie algebras

$$\mathcal{U}\colon T_{\text{poly}} \to D_{\text{poly}},$$

with explicit local formulas depending on integrals over configurations of points and expressed in terms of graphs. This results solves the deformation quantization problem by establishing a correspondence between formal Poisson structures and star products (bijective up to gauge equivalence).

Kontsevich's result, however, ignores the richer structures existent on T_{poly} and D_{poly}. Let now M be an oriented D-dimensional manifold with a fixed volume form ω. The pull back of the de Rham differential via contraction with ω endows the space T_{poly} with the structure of a BV algebra. On the other hand, there is a natural action of the cyclic group of order $n+1$ on D^n_{poly} given by "integration by parts" which, after the cyclic Deligne's conjecture (see [18]), induces a natural BV_∞ algebra structure on D_{poly}. The natural question to ask is whether Kontsevich's formality morphism can be extended to a BV_∞ quasi-isomorphism. Tamarkin [26,16] constructed a non-explicit Ger_∞ (homotopy Gerstenhaber) quasi-isomorphism $T_{\text{poly}} \to D_{\text{poly}}$ depending on a solution of Deligne's conjecture whose underlying Lie_∞ morphism was later shown by Willwacher [31] to be homotopy equivalent to Kontsevich's map if one uses the Alekseev–Torossian associator to construct a solution to Deligne's conjecture. Furthermore, Willwacher shows that the original formality morphism can be strictly extended to a Ger_∞ morphism. The full extension to the BV setting was given by the first author [2] who constructed a BV_∞ quasi-isomorphism $T_{\text{poly}} \to D_{\text{poly}}$ with explicit local formulas depending on integrals over configurations of framed points. One advantage of incorporating these richer structures into the discussion is that we may now view the algebraic operations as being parametrized by geometric objects, namely by the moduli spaces of genus zero surfaces with parametrized boundary components.

The subspace of cyclic invariants of D_{poly}, denoted by $D^\sigma_{\text{poly}} := \bigoplus_{n\geq 0} (D^n_{\text{poly}})^{\mathbb{Z}_{n+1}}$, is preserved by the Lie bracket and the Hochschild differential. The differential graded Lie algebra D^σ_{poly} is associated to a different deformation problem, namely the construction of *closed star products*. This led to the conjecture of an analogous formality statement, the "cyclic formality conjecture" [25]. Let $\text{div}_\omega\colon T^\bullet_{\text{poly}} \to T^{\bullet-1}_{\text{poly}}$ be the divergence operator