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Strict n-categories and augmented directed complexes model homotopy types

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ABSTRACT

In this paper we show that both the homotopy category of strict n-categories, $1 \leqslant n \leqslant \infty$, and the homotopy category of Steiner's augmented directed complexes are equivalent to the category of homotopy types. In order to do so, we first prove a general result, based on a strategy of Fritsch and Latch, giving sufficient conditions for a nerve functor with values in simplicial sets to induce an equivalence at the level of homotopy categories. We then apply this result to strict n-categories and augmented directed complexes, where the assumption of our theorem was first established by Ara and Maltsiniotis and of which we give an independent proof.

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Introduction

The homotopy theory of small categories was born with the introduction by Grothendieck of the nerve functor in [6], allowing the definition of weak equivalences in $\mathcal{C}at$, the category of small categories: a functor is a weak equivalence precisely when its nerve is a simplicial weak equivalence.

The first striking result of this theory appears in Illusie's thesis [8] (who credits it to Quillen) and states that the induced nerve functor induces an equivalence at the level of the homotopy categories. A careful study of the subtle homotopy theory of small categories by Thomason [12] led him to show another important result: the existence of a model structure on $\mathcal{C}at$ which is Quillen equivalent to the Kan–Quillen model structure on simplicial sets. This implies that small categories and simplicial sets are not only equivalent as homotopy categories, but actually as $(\infty, 1)$-categories.

Similarly Chiche [4], using ideas of del Hoyo [7], has shown that strict 2-categories model homotopy types. Moreover, Ara and Maltsiniotis [1] provided an abstract framework in which a Quillen equivalence between strict n-categories and simplicial sets can be established, using the Street nerve, as long as some conditions are satisfied; they also showed that these conditions hold for small 2-categories. One of these conditions, which they called condition (e), states that the unit morphism of the adjunction given by the Street nerve is a weak equivalence when evaluated at the nerve of a poset. In a subsequent paper [2] they proved in full generality this condition (e), making an important use of the augmented directed complexes introduced by Steiner [10], which act as a linearisation of ∞-categories.

In the same article, they also conjectured that for $1 \leqslant n \leqslant \infty$ the Street nerve induces an equivalence of homotopy categories between strict n-categories and simplicial sets, and that the same holds for augmented directed complexes with their own nerve functor. The aim of this paper is to prove these conjectures.

We consider a general framework with a nerve $U \colon \mathcal{C} \to \widehat{\mathbf{\Delta}}$ induced as a right adjoint of a left Kan extension F of a cosimplicial object of $\mathcal{C}$ along the Yoneda embedding of $\mathbf{\Delta}$, where $\mathcal{C}$ is a cocomplete category, $\mathbf{\Delta}$ is the category of simplices and $\widehat{\mathbf{\Delta}}$ is the category of simplicial sets. Using the properties of simplicial subdivision and adapting a strategy developed in an article by Fritsch and Latch [5], a short proof that U induces an equivalence at the level of homotopy categories can be given, as long as it satisfies an appropriate version of condition (e). This condition says that the unit $\eta \colon 1_{\widehat{\mathbf{\Delta}}} \to UF$ is a weak equivalence on the nerves of posets, that is for every poset E, the morphism of simplicial sets $N_1(E) \to UF(N_1(E))$ is a weak equivalence, where $N_1 \colon \mathcal{C}at \to \widehat{\mathbf{\Delta}}$ denotes the usual nerve. As a first example, we give a short proof of the well-known result stating that (ordered) simplicial complexes model homotopy types. More interestingly, Ara and Maltsiniotis [2] have established condition (e) for both the nerve $N_n \colon n\text{-}\mathcal{C}at \to \widehat{\mathbf{\Delta}}$ of strict n-categories, $1 \leqslant n \leqslant \infty$, and the nerve $\mathsf{N} \colon \mathcal{C}_{\mathrm{da}} \to \widehat{\mathbf{\Delta}}$ of augmented directed complexes. We can thus apply our theorem to these cases and get a proof of the above conjectures. Nevertheless, we give an independent proof of the condi-