Advances in Mathematics 331 (2018) 589–626

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Geometry of free loci and factorization of noncommutative polynomials

J. William Helton [a,1], Igor Klep [b,*,2], Jurij Volčič [c,3]

[a] Department of Mathematics, University of California San Diego, United States of America
[b] Department of Mathematics, The University of Auckland, New Zealand
[c] Department of Mathematics, Ben-Gurion University of the Negev, Israel

ARTICLE INFO

Article history:
Received 6 September 2017
Received in revised form 8 March 2018
Accepted 2 April 2018
Communicated by Petter Brändén

MSC:
primary 13J30, 15A22, 47A56
secondary 14P10, 16U30, 16R30

Keywords:
Noncommutative polynomial
Factorization
Singularity locus
Linear matrix inequality
Spectrahedron
Invariant theory

ABSTRACT

The free singularity locus of a noncommutative polynomial f is defined to be the sequence of hypersurfaces $\mathscr{Z}_n(f) = \{X \in \mathrm{M}_n(\Bbbk)^g : \det f(X) = 0\}$. The main theorem of this article shows that f is irreducible if and only if $\mathscr{Z}_n(f)$ is eventually irreducible. A key step in the proof is an irreducibility result for linear pencils. Arising from this is a free singularity locus Nullstellensatz for noncommutative polynomials. Apart from consequences to factorization in a free algebra, the paper also discusses its applications to invariant subspaces in perturbation theory and linear matrix inequalities in real algebraic geometry.

* Corresponding author.
E-mail addresses: helton@math.ucsd.edu (J. William Helton), igor.klep@auckland.ac.nz (I. Klep), volcic@post.bgu.ac.il (J. Volčič).
[1] Research supported by the NSF grant DMS 1500835. The author was supported through the program “Research in Pairs” (RiP) by the Mathematisches Forschungsinstitut Oberwolfach (MFO) in 2017.
[2] Supported by the Marsden Fund Council of the Royal Society of New Zealand. Partially supported by the Slovenian Research Agency grants P1-0222, L1-6722, J1-8132. The author was supported through the program “Research in Pairs” (RiP) by the Mathematisches Forschungsinstitut Oberwolfach (MFO) in 2017.
[3] Supported by The University of Auckland Doctoral Scholarship.

https://doi.org/10.1016/j.aim.2018.04.007

Contents

1. Introduction . . . 590
2. Preliminaries . . . 594
   2.1. Free loci of matrix pencils . . . 594
   2.2. Simultaneous conjugation of matrices . . . 595
3. Determinant of an irreducible pencil . . . 596
   3.1. Degree growth . . . 596
   3.2. Eventual irreducibility . . . 598
   3.3. Irreducible free loci . . . 601
4. Irreducibility of matrices over a free algebra . . . 602
5. Pencils with polynomial free locus and matrix perturbations . . . 606
   5.1. Perturbations and invariant subspaces . . . 606
   5.2. State space realizations . . . 606
   5.3. Flip-poly pencils . . . 608
   5.4. Minimal factorizations . . . 610
      5.4.1. Proof of Theorem 5.1 . . . 612
   5.5. A factorization result with missing variables . . . 614
6. Algorithms . . . 615
   6.1. Comparing polynomial free loci . . . 615
   6.2. Factorization via state space realizations . . . 616
7. Smooth points on a free locus . . . 618
8. Applications to real algebraic geometry . . . 621
   8.1. Boundaries of free spectrahedra . . . 621
   8.2. Randstellensatz . . . 624
Acknowledgments . . . 624
References . . . 624

1. Introduction

Algebraic sets, as zero sets of commutative polynomials are called, are basic objects in algebraic geometry and commutative algebra. One of the most fundamental results is Hilbert's Nullstellensatz, describing polynomials vanishing on an algebraic set. A simple special case of it is the following: if a polynomial h vanishes on a hypersurface given as the zero set of an irreducible polynomial f, then f divides h. Various far-reaching noncommutative versions of algebraic sets and corresponding Nullstellensätze have been introduced and studied by several authors [3,49,44,46]. Heavily reliant on these ideas and results are emerging areas of free real algebraic geometry [19,27] and free analysis [41,31,1,33]. In the free context there are several natural choices for the "zero set" of a noncommutative polynomial f. For instance, Amitsur proved a Nullstellensatz for the set of tuples of matrices X satisfying $f(X) = 0$ [3], and a conclusion for pairs (X, v) of matrix tuples X and nonzero vectors v such that $f(X)v = 0$ was given by Bergman [28]. In contrast with the successes in the preceding two setups, a Nullstellensatz-type analysis for the set of matrix tuples X making $f(X)$ singular (not invertible), which we call the *free singularity locus* of f (free locus for short), is much less developed. In this paper we rectify this. One of our main results connects free loci with factorization in free algebra [17,6,4,10,47] in the sense of the special case of Hilbert's Nullstellensatz mentioned above.