

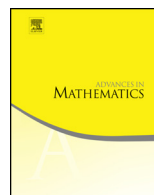


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Geometry of free loci and factorization of noncommutative polynomials



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ABSTRACT

The free singularity locus of a noncommutative polynomial f is defined to be the sequence of hypersurfaces $\mathcal{Z}_n(f) = \{X \in M_n(\mathbb{k})^g : \det f(X) = 0\}$. The main theorem of this article shows that f is irreducible if and only if $\mathcal{Z}_n(f)$ is eventually irreducible. A key step in the proof is an irreducibility result for linear pencils. Arising from this is a free singularity locus Nullstellensatz for noncommutative polynomials. Apart from consequences to factorization in a free algebra, the paper also discusses its applications to invariant subspaces in perturbation theory and linear matrix inequalities in real algebraic geometry.

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1. Introduction

Algebraic sets, as zero sets of commutative polynomials are called, are basic objects in algebraic geometry and commutative algebra. One of the most fundamental results is Hilbert’s Nullstellensatz, describing polynomials vanishing on an algebraic set. A simple special case of it is the following: if a polynomial h vanishes on a hypersurface given as the zero set of an irreducible polynomial f , then f divides h . Various far-reaching noncommutative versions of algebraic sets and corresponding Nullstellensätze have been introduced and studied by several authors [3,49,44,46]. Heavily reliant on these ideas and results are emerging areas of free real algebraic geometry [19,27] and free analysis [41,31,1,33]. In the free context there are several natural choices for the “zero set” of a noncommutative polynomial f . For instance, Amitsur proved a Nullstellensatz for the set of tuples of matrices X satisfying $f(X) = 0$ [3], and a conclusion for pairs (X, v) of matrix tuples X and nonzero vectors v such that $f(X)v = 0$ was given by Bergman [28]. In contrast with the successes in the preceding two setups, a Nullstellensatz-type analysis for the set of matrix tuples X making $f(X)$ singular (not invertible), which we call the *free singularity locus* of f (free locus for short), is much less developed. In this paper we rectify this. One of our main results connects free loci with factorization in free algebra [17,6,4,10,47] in the sense of the special case of Hilbert’s Nullstellensatz mentioned above.

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